Generalized Matrix Graphs and Completely Independent Critical Cliques

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Abstract

A \( k \)-dimensional \( n \)-square matrix is defined and certain properties of such matrices are investigated. Two particular graph constructions involving \( k \)-dimensional \( n \)-square matrices are given and the graphs so constructed are called matrix graphs. Properties of matrix graphs are determined and an application of matrix graphs to completely independent critical clique is provided. Some attention is given to this application and its relationship with the double-critical conjecture that the only vertex double-critical graph is the complete graph.

Keywords and phrases: matrix graph, chromatic number, critical clique, \( k \)-matching, completely independent critical cliques, double-critical conjecture

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1 Introduction and notation

In the late 1940’s, G. A. Dirac defined critical graphs for the purpose of simplifying the central problems in the theory of graph coloring. Several results on critical graphs containing few edges have been established; e.g., [4], [5], [10], and [11]. Likewise, several results on critical graphs containing many edges have been established; e.g., [15] and [7]. However, determining bounds on the number of edges in critical graphs is not the objective of this paper. Rather, relations between particular sets of critical vertices will be investigated. This paper is a continuation and generalization of the results obtained in [12].

Most of the notation and terminology follows that found in [2] and [3]. The graphs considered in this paper are finite, undirected, and simple. For a given graph \( G \), the vertex set of \( G \) and the edge set of \( G \) are denoted by \( V (G) \) and \( E (G) \), respectively. The order of \( G \) is the cardinality of \( V (G) \) and is denoted by \( |V (G)| \). The complete graph having order \( r \) shall be denoted by \( K_r \). An \( r \)-clique of \( G \) is a subgraph \( K \) of \( G \) isomorphic to \( K_r \). A subset \( I \) of \( V (G) \) is said to be independent whenever no two distinct vertices in \( I \) are adjacent. The maximum cardinality of an independent subset of \( V (G) \) is denoted by \( \alpha (G) \). A subset \( M \) of \( E (G) \) is said to be independent whenever no two edges in \( M \) share a common vertex; an independent subset \( M \) of \( E (G) \) is often referred to as a matching. A matching \( M \) is called a \( k \)-matching whenever \( |M| = k \). For a subset \( X \) of \( V (G) \), the subgraph of \( G \) induced by \( X \) is denoted by \( G [X] \). All vertex colorings considered will be proper; i.e., a partition of \( V (G) \) into independent subsets of \( V (G) \) called color
classes. Lastly, the minimum cardinality of a partition of $V(G)$ admitted by a proper vertex coloring of $G$ is called the chromatic number of $G$ and is denoted by $\chi(G)$.

Let $\mathbb{N} = \{1, 2, 3, \ldots \}$ be the set of natural numbers. For $n \in \mathbb{N}$, the set $S_n$ denotes the set of all permutations on $n$ elements. The general element $\sigma$ in $S_n$ will be written as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. When convenient, $\sigma$ may also be written as a formal string, $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$. If $\tau \in S_n$ with $\tau = \tau_1 \tau_2 \cdots \tau_n$, then $\hat{\tau}(i)$ denotes the formal substring of $\tau$ determined by deleting the $i$th character of $\tau$. More precisely,

$$\hat{\tau}(i) = \begin{cases} 
\tau_i \tau_{i+1} \cdots \tau_n, & \text{if } i = 1 \\
\tau_1 \tau_2 \cdots \tau_{i-1}, & \text{if } 1 < i < n \\
\tau_1 \tau_2 \cdots \tau_{n-1}, & \text{if } i = n.
\end{cases}$$

The $j$th character of the formal string $\hat{\tau}(i)$, for $1 \leq j \leq n - 1$, will be denoted by $\hat{\tau}_j(i)$ and is given by

$$\hat{\tau}_j(i) = \begin{cases} 
\tau_j, & \text{if } j < i \\
\tau_{j+1}, & \text{if } j \geq i.
\end{cases}$$

Let $A$ and $B$ be sets. The symmetric difference of $A$ and $B$, denoted by $A \Delta B$, is the set $(A - B) \cup (B - A)$. Let $S = \{ S_\omega : \omega \in \Omega \}$ be an indexed family of sets. The generalized union of the indexed family is denoted by $\cup S$ and is given by

$$\cup S = \bigcup_{\omega \in \Omega} S_\omega.$$  

For $n \in \mathbb{N}$, an arbitrary Latin square of order $n$ will be denoted by $L_n$. The $i$th row of a general Latin square $L_n$, where $1 \leq i \leq n$, will be denoted by $\Lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$. As usual, $\delta_{ij}$ will denote the Kronecker delta function. Lastly, let

$$M = \{ a_1 \cdot x_1, a_2 \cdot x_2, \ldots, a_m \cdot x_m \}$$

be a (finite) multiset having $m$ distinct elements $x_1, x_2, \ldots, x_m$. Here, the natural numbers $a_1, a_2, \ldots, a_m$ are called the repetition numbers and denote the number of times the corresponding element appears in the multiset $M$. Hence, it is clear that

$$|M| = \sum_{i=1}^{m} a_i.$$  

A submultiset of $M$ is a set

$$T = \{ s_1 \cdot x_1, s_2 \cdot x_2, \ldots, s_m \cdot x_m \}$$

satisfying $0 \leq s_i \leq a_i$ for $i \in I_m$. An $r$-submultiset satisfies

$$|T| = \sum_{i=1}^{m} s_i = r$$

and may also be referred to as an $r$-combination.

2 Generalized k-dimensional n-square matrices

2.1 Terminology and basic definitions

For $n \in \mathbb{N}$, let $I_n$ represent the $n$th segment of $\mathbb{N}$, that is, $I_n = \{1, 2, \ldots, n\}$. Also, for $k, n \in \mathbb{N}$, let $I_n^k$ be the $k$-fold Cartesian product of $I_n$:

$$I_n^k = \prod_{i=1}^{k} I_n = I_n \times I_n \times \cdots \times I_n.$$
The general element of $I_n^k$ will be denoted by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. When convenient, $\alpha$ may also be viewed as a function, $\alpha : I_k \to I_n$, defined by the rule $\alpha(i) = \alpha_i$, and written as a formal string $\alpha = \alpha_1\alpha_2\cdots\alpha_k$. Moreover, we adopt the conventions that $I_n^1 = \{1\}$ and $I_n^n = I_n$. For $T \subseteq \mathbb{N}$ and $c \in \mathbb{N}$, the scalar product $cT$ is defined by $cT = \{ct : t \in T\}$.

**Definition 1** Let $S$ be any nonempty set. A $k$-dimensional $n$-square matrix $A$ over $S$ is any function $A : I_n^k \to S$. Let $\mathcal{M}^k_n$ be the set of all $k$-dimensional $n$-square matrices.

For the purposes of this paper, the set $S$ of Definition 1 will play no role. Therefore, specific mention of $S$ will be omitted and we will simply say that $A$ is a $k$-dimensional $n$-square matrix. Furthermore, Definition 1 imposes no conditions on $k$ and $n$. However, we will require, except for one instance in Section 4, that $n \geq k$. The usual matrix notation will be adopted. That is, the general entry of $A \in \mathcal{M}^k_n$ will be denoted by $a_{\alpha}$, where $\alpha \in I_n^k$. Also, $A$ will be written as $A = [a_{\alpha}]$. For $A \in \mathcal{M}^k_n$, let $A^*$ be the underlying set consisting of the entries of $A$; i.e., $A^* = \{a_{\alpha} : \alpha \in I_n^k\}$.

**Definition 2** Let $A \in \mathcal{M}^k_n$ and let $X$ be a nonempty subset of $I_n^k$. The entries of $A$ corresponding to $X$ will be denoted by $X_A$. Precisely,

$$X_A = \{a_{\alpha} \in A^* : \alpha \in X\}.$$

Similarly, let $Y$ be a nonempty subset of $A^*$. The indices of $I_n^k$ corresponding to $Y$ will be denoted by $Y_{I_n^k}$. Precisely,

$$Y_{I_n^k} = \{\alpha \in I_n^k : a_{\alpha} \in Y\}.$$

With a slight abuse of notation, we will write $X^* = X_A$ even though $X$ is not necessarily an element of $\mathcal{M}^k_n$. However, this should cause no confusion.

**Definition 3** Let $A \in \mathcal{M}^k_n$ and let $X$ be a nonempty subset of $I_n^k$. Also, let $k' \in \mathbb{N}$ with $1 \leq k' \leq k$. Any matrix $B \in \mathcal{M}^{k'}_n$ such that $X^* = X_A$ can be placed into a one-to-one correspondence with $B^*$ is called a matrix associated with $X$.

It is clear from Definition 3 that there is a matrix $B$ associated with $X$ if and only if $|X| = n^{k'}$. Various subsets of $A^*$ will play an important role. They will be used often enough to warrant some additional definitions.

**Definition 4** Let $A \in \mathcal{M}^k_n$. The main diagonal of $A$, denoted by $\Delta(A)$, is the set

$$\Delta(A) = \{a_{\alpha} \in A^* : \alpha = c(1, 1, \ldots, 1) \text{ for some } c \in I_n\}.$$

**Definition 5** Let $A \in \mathcal{M}^k_n$. For $i \in I_k$ and $j \in I_n$, the $j$th face of $A$ in the $i$th direction, denoted by $F_{ij}(A)$, is the set

$$F_{ij}(A) = \{a_{\alpha} \in A^* : \alpha_i = j\}.$$

Depending on the situation, it is more advantageous to consider not the definition of the various subsets of $A^*$ in terms of imposed conditions on the coordinates of the entries of $A$, but rather in terms of the Cartesian product of subsets of $I_n$. In this regard, the face $F_{ij}(A)$ can equivalently be viewed as the restriction of $A$ to the set

$$X_{ij} = \prod_{t=1}^{k} j^{\delta_{it}} I_n^{1-\delta_{it}}.$$

In other words, $F_{ij}(A) = X_{ij}^*$. Additionally, it will be important at times to consider a particular face containing the entry $a_{\alpha}$ of a $k$-dimensional $n$-square matrix $A$. This leads to the following definition.
Definition 6 Let $A \in \mathcal{M}_n^k$. For $a_\alpha \in A^*$, the $i$th face of $A$ at $a_\alpha$, denoted by $F_i(a_\alpha)$, is the set

$$F_i(a_\alpha) = \{a_\beta \in A^* : \alpha_i = \beta_i\}.$$ 

Also, the face set of $A$ at $a_\alpha$, denoted by $\mathcal{F}_{a_\alpha}$, is the set

$$\mathcal{F}_{a_\alpha} = \{F_i(a_\alpha) : i \in I_k\}.$$

There is an intimate connection between Definition 5 and Definition 6. Precisely, the face $F_i(a_\alpha)$ can be viewed as the $\alpha_i$-th face of $A$ in the $i$th direction; i.e., $F_i(a_\alpha) = F_i(a_\alpha_i)(A)$. This connection will be exploited often. Moreover, observe that $F_i(a_\alpha) = F_i(a_\beta)$ if and only if $\alpha_i = \beta_i$.

Remark 1 The figures in this article may contain several highlighted vertices either to illustrate certain sets of vertices or to illustrate certain adjacencies that exist between vertices. The highlighted vertices do not necessarily indicate a proper coloring of a subset of the vertices. To avoid any confusion, it will be stated explicitly when the highlighted vertices represent a proper coloring.

Example 1 In Figures 1 - 3 below, the faces $F_{12}(A) = F_1(a_{231})$, $F_{23}(A) = F_2(a_{231})$, and $F_{31}(A) = F_3(a_{231})$ are illustrated for a 3-dimensional 4-square matrix $A$, respectively. The elements contained in the faces are highlighted in either red or blue. The element colored blue is the element at which the face is located. The light gray lines in the grid graphs are for visual purposes only and are not meant to indicate edges in a graph.

![Figure 1: The face $F_{12}(A) = F_1(a_{231})$](image-url)
In addition to the faces of a $k$-dimensional $n$-square matrix $A$, it is important to consider sets of entries that are "perpendicular" to the faces. In the same spirit as the normal line at a point is perpendicular to the tangent line at a point, the notion of a "normal" of a $k$-dimensional $n$-square matrix is defined next.
Definition 7 Let $A \in \mathcal{M}_n^k$, also, let $F_i = F_i(a_\alpha)$ be the $i$th face of $A$ at $a_\alpha$. The $i$th normal of $A$ at $a_\alpha$, denoted by $\eta^i_{F_i}(a_\alpha)$, is the set

$$\eta^i_{F_i}(a_\alpha) = \{ a_\beta \in A^* : \beta_j = \alpha_j \text{ for all } j \neq i \}.$$ 

Also, the normal set of $A$ at $a_\alpha$, denoted by $\eta^i(a_\alpha)$, is the set

$$\eta^i(a_\alpha) = \{ \eta^i_{F_i}(a_\alpha) : i \in I_k \}.$$ 

It is not difficult to verify that the $i$th normal of $A$ at $a_\alpha$ can be expressed in terms of the faces of $A$ at $a_\alpha$. In particular,

$$\eta^i_{F_i}(a_\alpha) = \bigcap_{t \neq i} F_t(a_\alpha).$$

Definition 7 is clarified by the next example.

Example 2 Figure 4 below shows the union of the normal set of a matrix $A \in \mathcal{M}_4^3$ at $a_{231}$. Note that the 1st normal of $A$ at $a_{231}$ is perpendicular to $F_1(a_{231})$, the 2nd normal of $A$ at $a_{231}$ is perpendicular to $F_2(a_{231})$, and the 3rd normal of $A$ at $a_{231}$ is perpendicular to $F_3(a_{231})$. The green and light gray lines are for visual purposes only as stated earlier. Also, the vertex at which the normal set located is indicated in blue and the remaining vertices in the normal set are highlighted in red.

![Figure 4: The normal set of $A$ at $a_{231}$.](image)

The next two definitions will prove useful in Section 3.1.

Definition 8 Let $A \in \mathcal{M}_n^k$, where $k \geq 3$, and let $a_\alpha \in A^*$. Further let $i_1, i_2 \in I_k$, with $i_1 \neq i_2$. The $(i_1, i_2)$-hyperface of $A$ at $a_\alpha$, denoted by $F_{i_1}^{i_2}(a_\alpha)$, is the set

$$F_{i_1}^{i_2}(a_\alpha) = \{ a_\beta \in A^* : \beta_{i_1} = \alpha_{i_1} \text{ and } \beta_{i_2} = \alpha_{i_2} \}.$$ 

Also, the hyperface set of $A$ at $a_\alpha$, denoted by $\mathcal{F}^{a_\alpha}$, is the set

$$\mathcal{F}^{a_\alpha} = \{ F_{i_1}^{i_2}(a_\alpha) : i_1, i_2 \in I_k \text{ and } i_1 \neq i_2 \}.$$
In a similar fashion, the notion of a hypernormal is defined.

**Definition 9** Let \( A \in \mathcal{M}_n^k \), where \( k \geq 3 \). Also, let \( F_{i_1} (a_\alpha) \) be the \( i_1 \)-st face of \( A \) at \( a_\alpha \). For \( i_2 \in I_k \) with \( i_2 \neq i_1 \), the \((i_1, i_2)\)-hypernormal of \( A \) at \( a_\alpha \) is the set

\[
\eta_{F_{i_1}^{i_2}} (a_\alpha) = \{ a_{ij} \in A^*: \beta_j = \alpha_j \text{ for all } j \neq i_1, i_2 \}.
\]

In three dimensions, the notions of hyperface and normal are equivalent as are the notions of face and hypernormal. Additionally, the notions of faces and normals, as well as hyperfaces and hypernormals, can be expressed in terms of Cartesian products. For instance, the hypernormal \( \eta_{F_{i_1}^{i_2}} (a_\alpha) \) can be equivalently viewed as the restriction of \( A \) to the set

\[
X_{i_1, i_2} (a_\alpha) = \prod_{t=1}^k (\alpha_t)^{1-\delta_{i_1 t}-\delta_{i_2 t}} I_n^{\delta_{i_1 t}+\delta_{i_2 t}}.
\]

In other words, \( \eta_{F_{i_1}^{i_2}} (a_\alpha) = X_{i_1, i_2} (a_\alpha) \). Also, the hyperface \( F_{i_1}^{i_2} (a_\alpha) \) can be viewed as the restriction of \( A \) to the set

\[
Y_{i_1, i_2} (a_\alpha) = \prod_{t=1}^k (\alpha_{i_1 t})^{\delta_{i_1 t}} (\alpha_{i_2 t})^{\delta_{i_2 t}} I_n^{1-\delta_{i_1 t}-\delta_{i_2 t}}.
\]

Thus, \( F_{i_1}^{i_2} (a_\alpha) = Y_{i_1, i_2} (a_\alpha) \).

**Definition 10** Let \( A \in \mathcal{M}_n^k \) and let \( a_\alpha \in A^* \). The \( k \)-dimensional \((n - 1)\)-square submatrix of \( A \) determined by \( a_\alpha \), denoted by \( A (\lfloor x_{\alpha} \rfloor) \), is the restriction of \( A \) to the set

\[
I_n^k (a_\alpha) = \prod_{i=1}^k (I_n \setminus \{ \alpha_i \})
\]

Furthermore, \( A (\lfloor x_{\alpha} \rfloor) \) will be written in expanded form as

\[
A (\lfloor x_{\alpha} \rfloor) = A (F_1 (a_\alpha) | F_2 (a_\alpha) | \cdots | F_k (a_\alpha))
\]

**Definition 11** Let \( i \in I_k \). The \( i \)-th projection of the set \( I_n^k \), denoted by \( \pi_i (I_n^k) \), is the set

\[
\pi_i (I_n^k) = \prod_{j=1}^k I_n^{\delta_{i j}}
\]

**Definition 12** Let \( S \) be any nonempty set. For \( i \in I_k \), a \( k \)-dimensional \( n \)-row matrix is any function \( X_i : \pi_i (I_n^k) \to S \). For \( \alpha \in \pi_i (I_n^k) \), where \( \alpha_j = \alpha_{i j} \), for \( j \in I_k \), we write

\[
X_i (\alpha) = \begin{cases} x_i^\alpha \\ = x_i^a \\ = x_i^{1, 1, \ldots, a_i, \ldots, 1, 1} \end{cases}
\]

and

\[
X_i = \left[ x_i^\alpha \right]
\]

\[
= \left[ x_i^{1, 1, \ldots, 1, 1, 1}, x_i^{1, 1, \ldots, 2, 1, 1}, \ldots, x_i^{1, 1, \ldots, n, 1, 1} \right],
\]

where the "centered" coordinate is in the \( i \)-th position.
2.2 On the ordering of entries in a $k$-dimensional $n$-square matrix

Recall that the domain of a $k$-dimensional $n$-square matrix is the Cartesian product $I_n^k$. The set $I_n^k$ is ordered in the usual way. For $\alpha, \beta \in I_n^k$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$, we have $\alpha < \beta$ if and only if $\alpha_1 < \beta_1$ or there is an index $m \in I_{k-1}$ such that $\alpha_m = \beta_m$ for $i \in I_{m}$ and $\alpha_{m+1} < \beta_{m+1}$. Also, $\alpha = \beta$ if and only if $\alpha_i = \beta_i$ for $i \in I_k$. Therefore, the ordering of the elements of a $k$-dimensional $n$-square matrix will correspond to the natural [dictionary] ordering of its domain. Moreover, any subset of a $k$-dimensional $n$-square matrix will be ordered in the same fashion.

2.3 On the cardinalities of subsets of a $k$-dimensional $n$-square matrix

In Section 2.1, various subsets of a matrix $A \in M_n^k$ were defined. Cardinalities of these various subsets are determined next.

**Proposition 1** Let $A \in M_n^k$. The cardinality of any normal of $A$ is $n$.

**Proposition 2** Let $A \in M_n^k$ and let $a_\alpha \in A^*$. Then

$$|\bigcup \eta^\perp (a_\alpha)| = k (n - 1) + 1.$$ 

**Proof.** Recall the normal set of $A$ at $a_\alpha$ is given by

$$\eta^\perp (a_\alpha) \{ \eta^\perp (a_\alpha) : i \in I_k \}.$$ 

For $i_1, i_2 \in I_k$, with $i_1 \neq i_2$, observe that $\eta^\perp (a_\alpha) \cap \eta^\perp (a_\alpha) = \{a_\alpha\}$. Therefore, by Proposition 1, it follows that

$$|\bigcup_{i=1}^{k} \eta^\perp (a_\alpha) \setminus \{a_\alpha\}| = \sum_{i=1}^{k} (n - 1) = k (n - 1)$$

Consequently, $|\bigcup \eta^\perp (a_\alpha)| = k (n - 1) + 1$.

We remark that the result of Proposition 2 can be obtained also by applying the principle of inclusion-exclusion. For,

$$|\bigcup \eta^\perp (a_\alpha)| = \left| \bigcup_{i=1}^{k} \eta^\perp (a_\alpha) \right| = \sum_{i=1}^{k} |\eta^\perp (a_\alpha)| - \sum_{i=2}^{k} (-1)^{i} \binom{k}{i} = nk - (k - 1) = k (n - 1) + 1.$$ 

**Proposition 3** Let $A \in M_n^k$. The cardinality of any face of $A$ is $n^{k-1}$. 

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Proposition 4  Let $A \in \mathcal{M}_n^k$ and let $a_\alpha \in A^*$. Then

$$| \cup \mathcal{F}_{a_\alpha} | = kn^{k-1} - \sum_{i=2}^{k} (-1)^i \binom{k}{i} n^{k-i}.$$  

**Proof.** Recall that the face set of $A$ at $a_\alpha$ is

$$\mathcal{F}_{a_\alpha} = \{ F_i(a_\alpha) : i \in I_k \}.$$  

For $i_1, i_2 \in I_k$, with $i_1 \neq i_2$, observe that

$$F_{i_1}(a_\alpha) \cap F_{i_2}(a_\alpha) = \{ a_\beta \in A^* : \alpha_{i_1} = \beta_{i_1} \text{ and } \alpha_{i_2} = \beta_{i_2} \},$$

the $(i_1, i_2)$-hyperface of $A$ at $a_\alpha$. Therefore,

$$| F_{i_1}(a_\alpha) \cap F_{i_2}(a_\alpha) | = n^{k-2}.$$  

More generally, for $i_{s_1}, i_{s_2} \in I_k$, with $i_{s_1} \neq i_{s_2}$, observe that

$$\bigcap_{s=1}^{t} F_{i_s}(a_\alpha) = n^{k-t}.$$  

Hence, by applying Proposition 3 and the principle of inclusion-exclusion, we find that

$$| \cup \mathcal{F}_{a_\alpha} | = \left| \bigcup_{i=1}^{k} F_i(a_\alpha) \right| = \sum_{i=1}^{k} | F_i(a_\alpha) | - \sum_{i=2}^{k} (-1)^i \binom{k}{i} n^{k-i} = kn^{k-1} - \sum_{i=2}^{k} (-1)^i \binom{k}{i} n^{k-i}.$$  

This completes the proof.  

3 Matrix graphs

In the previous section, the notion of a $k$-dimensional $n$-square matrix has been introduced and some of the basic properties of these matrices have be investigated. In the current section, the notion of a $k$-dimensional $n$-square matrix is used to define a particular graph construction. A graph constructed in such a fashion is called a matrix graph. There are two primary ways to construct a graph from a given $k$-dimensional $n$-square matrix. For lack of better terminology, they will be referred to as a Type I matrix graph and a Type II matrix graph. Each of these constructions will be similar in nature; but, one construction, namely a Type I matrix graph, will prove to be more useful for our purposes. Therefore, more attention will be devoted to Type I matrix graphs.
3.1 Construction of Type I and Type II matrix graphs

The construction of Type I and Type II matrix graphs depend on the selection of a $k$-dimensional $n$-square matrix. Recall that we have imposed the condition that $n \geq k$.

**Type I Matrix Graph** Let $A = [a_{ij}] \in \mathcal{M}_n^k$. Denote a Type I matrix graph by $G_1(A)$. Define the vertex set of $G_1(A)$ by setting $V(G_1(A)) = A^*$ and by defining the edge set of $G_1(A)$ according to the following rule:

$$a_{i\alpha}a_{j\beta} \in E(G_1(A)) \text{ if and only if } a_{i\beta} \not\in \cup \mathcal{F}_{a_{\alpha}}.$$  

The graph $G_1(A)$ will be called a Type I matrix graph determined by $A$. We remark in a Type I matrix graph for $A \in \mathcal{M}_n^3$, that $a_{ij}a_{rs} \in E(G_1(A))$ if and only if $r \neq i$ and $s \neq j$. In other words, vertex $a_{ij}$ is adjacent to all vertices that remain when the $i$th row and $j$th column are deleted from the $n \times n$ matrix $A$. The definition of the edge set of a Type I matrix graph is the generalization of this notion. Figure 5 below shows all vertices (in red) adjacent to vertex $a_{221}$ (in blue) in a Type I matrix graph for a matrix $A \in \mathcal{M}_3^3$. The green lines demonstrate the adjacencies and the light gray lines are for visual purposes only. Note that some edges (in green) may overlap. Recall as in Remark 1, we are not indicating a coloration of the vertices.

![Figure 5: Vertices adjacent to vertex $a_{221}$](image)

The next two illustrations represent the entire Type I matrix graph for $A \in \mathcal{M}_3^3$. The visualizations in Figure 6 and Figure 7 below are the initially conceived matrix form and a circular embedding, respectively. Also illustrated in these two figures is a $3$-coloring of $G_1(A)$. 

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![Image of graph](image)
Figure 6: $G_1 (A)$ as originally conceived.

Figure 7: Circular embedding of $G_1 (A)$. 
**Type II Matrix Graph** Let $A = [a_{ij}] \in \mathcal{M}_n^k$. Denote a Type II matrix graph by $G_2(A)$. Define the vertex set of $G_2(A)$ by setting $V(G_2(A)) = A^*$ and by defining the edge set of $G_2(A)$ by the rule

$$a_\alpha a_\beta \in E(G_2(A)) \text{ if and only if } a_\beta \notin \cup \eta^\perp (a_\alpha).$$

The graph $G_2(A)$ will be called a Type II matrix graph determined by $A$. Figure 8 below shows all vertices (in red) adjacent to vertex $a_{231}$ (in blue) in a Type II matrix graph for a matrix $A \in \mathcal{M}_3^3$. The green lines demonstrate the adjacencies and the light gray lines are for visual purposes only. Note that some edges (in green) may overlap. Recall as in Remark 1, we are not indicating a coloration of the vertices.

The next two illustrations represent the entire Type II matrix graph for a matrix $A \in \mathcal{M}_3^3$. The visualizations in Figure 9 and Figure 10 below are the initially conceived matrix form and a circular embedding, respectively. Also illustrated in these two figures is a 9-coloring of $G_2(A)$.
Figure 9: \( G_2(A) \) as originally conceived.

Figure 10: Circular embedding of \( G_2(A) \).
3.2 On subgraphs of a Type I matrix graph

Consider $A \in \mathcal{M}^{k+1}_n$, where $k \geq 2$ and $n \geq k+1$. Let $G_1(A)$ be a Type I matrix graph. We would like to establish the existence of a subset $X^*$ of $A^*$ such that the induced subgraph $G_1(A)[X^*]$ is isomorphic to a Type I matrix graph determined by a matrix $C \in \mathcal{M}^k_n$. In this case, $C$ will be an associated matrix for $X^*_n$, the indices of $I_n^{k+1}$ corresponding to $X^*$. For convenience, this Type I matrix graph will be denoted by $H_1(C)$ so that $G_1(A)[X^*] \cong H_1(C)$. In fact, it will be shown that $V(G_1(A))$ can be partitioned into $n$ subsets, each one of which determines an induced subgraph of $G_1(A)$ that is isomorphic to $H_1(C)$.

**Proposition 5** Let $k \in \mathbb{N}$, where $k \geq 2$, and let $A \in \mathcal{M}^{k+1}_n$. The Type I matrix graph $G_1(A)$ contains an induced subgraph isomorphic to a Type I matrix graph $H_1(C)$ determined by a matrix $C \in \mathcal{M}^k_n$.

**Proof.** Let $k \in \mathbb{N}$ and let $A \in \mathcal{M}^{k+1}_n$, where $k \geq 2$. Consider a Type I matrix graph $G_1(A)$. Select an arbitrary permutation of $I_n$, say $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$, and also select an arbitrary 2-permutation of $I_k$, say $\rho = \rho_1 \rho_2$. The set

$$T(\lambda; \rho) = \left\{ \prod_{i=1}^{k+1} j^{\delta_{i\rho_1} \lambda_{i\rho_2}} I_n^{1-\delta_{i\rho_1} \rho_2} : j \in I_n \right\}$$

consists of $n$ subsets of $I_n^{k+1}$ each one of which corresponds to a hyperface of $A$, namely $F_{\rho_2}(a_\alpha)$ for any $a_\alpha$ satisfying $\alpha_{\rho_1} = j$ and $\alpha_{\rho_2} = \lambda_j$. Since two of the $k+1$ coordinate positions, namely $\rho_1$ and $\rho_2$, of each member of $T(\lambda; \rho)$ are fixed, it is clear that each member of $T(\lambda; \rho)$ is a nonempty subset of $I_n^{k+1}$ having cardinality $n^{k-1}$. Define $X = \bigcup T(\lambda; \rho)$. Then $|X| = n \cdot n^{k-1} = n^k$ and; moreover, $n \geq k+1 > k$. Hence, there exists a matrix $C \in \mathcal{M}^k_n$ associated with $X$. We claim that $G_1(A)[X^*] \cong H_1(C)$, where $H_1(C)$ is a Type I matrix graph determined by $C$. Let $F_{a_\alpha}^{X^*}$ be the face set of $a_\alpha$ in $X^*$. The claim follows immediately from the fact that for all $a_\alpha, a_\beta \in X^*$, we have $a_\beta \in F_{a_\alpha}^{X^*}$ if and only if $a_\beta \in F_{a_\alpha}$.

**Corollary 1** Let $k \in \mathbb{N}$, where $k \geq 2$, and let $G_1(A)$ be a Type I matrix graph, where $A \in \mathcal{M}^{k+1}_n$. There exists a Type I matrix graph $H_1(C)$, for some matrix $C \in \mathcal{M}^k_n$, and a partition $P$ of $V(G_1(A))$ such that $G_1(A)[P] \cong H_1(C)$ for each $P \in \mathcal{P}$.

**Proof.** We have demonstrated the existence of a subset of $A^*$ which admits an induced subgraph of $G_1(A)$ that is isomorphic to a Type I matrix graph having dimension one less than that of $G_1(A)$. It is now shown that $V(G_1(A))$ can be partitioned in such a way that each element of the partition admits an induced subgraph of $G_1(A)$ that is isomorphic to a Type I matrix graph having dimension one less than that of $G_1(A)$. Select an arbitrary Latin square having order $n$, say

$$L_n = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn}
\end{bmatrix} = \begin{bmatrix}
\Lambda_1 \\
\Lambda_2 \\
\vdots \\
\Lambda_n
\end{bmatrix},$$

where $\Lambda_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in})$ is the $i$th row of the Latin square $L_n$. For a fixed 2-permutation of $I_k$, say $\rho = \rho_1 \rho_2$, define

$$T(L_n; \rho) = \bigcup_{i=1}^{n} \{ T(\Lambda_i; \rho) \},$$

where, as in the proof Proposition 5 above, the set $T(\Lambda_i; \rho)$ is defined as

$$T(\Lambda_i; \rho) = \left\{ \prod_{i=1}^{k+1} j^{\delta_{i\rho_1} \lambda_{i\rho_2}} I_n^{1-\delta_{i\rho_1} \rho_2} : j \in I_n \right\}.$$
To see that the set $T(L_n; \rho)$ is a pairwise disjoint collection of subsets of $I_n^k$, suppose to the contrary that $\alpha \in T(A_q; \rho) \cap T(A_{q_2}; \rho)$, where $q_1, q_2 \in I_n$ with $q_1 \neq q_2$. By the definition of $T(A_q; \rho)$, the $(k+1)$-tuple $\alpha$ would be expressible in two ways as

$$\alpha = j_1^1 \lambda_{q_1,j_1} I_n^{1-\delta_{i_1}} - \delta_{i_2},$$

and

$$\alpha = j_2^1 \lambda_{q_2,j_2} I_n^{1-\delta_{i_1}} - \delta_{i_2},$$

for some $j_1, j_2 \in I_n$. Necessarily, $j_1 = j_2$ and $\lambda_{q_1,j_1} = \lambda_{q_2,j_2}$. But since $L_n$ is a Latin square, it would have to be that $q_1 = q_2$ contrary to the assumption that $q_1 \neq q_2$. This is a contradiction and confirms that $T(L_n, \rho)$ is a pairwise disjoint collection of nonempty subsets of $I_n^k$. Moreover,

$$|\cup T(L_n; \rho)| = n \cdot n^k = n^{k+1}$$

so that $\cup T(L_n; \rho) = I_n^{k+1}$. Finally, for $q \in I_n$, define $X_q = \cup T(A_q; \rho)$. By Proposition 5, we find that $G_1(A)[X_q] \equiv H_1(C)$, for some $C \in \mathcal{M}_n^k$. It follows that the collection

$$P = \{X_q : q \in I_n\}$$

is such a partition. ■

3.3 On independent subsets in a Type I matrix graph

First, a few general observations are noted regarding the adjacencies and nonadjacencies in a Type I matrix graph $G_1(A)$. Recall that the face set of $A$ at $a_\alpha$ is given by $\mathcal{F}_{a_\alpha} = \{F_i(a_\alpha) : i \in I_k\}$. Suppose that $F_{i_0}(a_\alpha) \in \mathcal{F}_{a_\alpha}$ and $a_\gamma \notin F_{i_0}(a_\alpha)$. Then there is an element $a_\beta$ of $A^*$ such that $a_\beta \in F_{i_0}(a_\alpha)$ and $a_\gamma \in \eta_{F_{i_0}}(a_\beta)$. Therefore, in $F_{i_0}(a_\alpha)$, the vertex $a_\gamma$ fails to be adjacent to each vertex in the set

$$\bigcup_{j \neq i_0} \eta_{F_j}^+(a_\beta).$$

By Proposition 1, note that

$$\left| \bigcup_{j \neq i_0} \eta_{F_j}^+(a_\beta) \right| = (k-1)(n-1) + 1.$$  

Moreover, $a_\gamma$ is adjacent to all other vertices in the face $F_{i_0}(a_\alpha)$. Consequently, $a_\gamma$ satisfies the following:

(i) $a_\gamma$ is adjacent to exactly $n^{k-1} - \left[(k-1)(n-1) + 1\right]$ vertices in $F_{i_0}(a_\alpha)$.

(ii) $a_\gamma$ is not adjacent to exactly $(k-1)(n-1) + 1$ vertices in $F_{i_0}(a_\alpha)$.

Let $A \in \mathcal{M}_n^k$ and consider a Type I matrix graph $G_1(A)$. We now consider certain independent subsets of $V(G_1(A))$. The next proposition is straightforward.

Proposition 6 The faces of a Type I matrix graph are independent.

**Proof.** Let $A \in \mathcal{M}_n^k$ and let $G_1(A)$ be a Type I matrix graph. Suppose that $a_\alpha \in A^*$ and consider an arbitrary face containing $a_\alpha$, say $F_{i_0}(a_\alpha)$ for some $i \in I_k$. If $a_\beta \in A^*$ and $a_\beta \in F_{i_0}(a_\alpha)$, then $a_\beta \notin \cup \mathcal{F}_{a_\alpha}$. Therefore, $a_\alpha a_\beta \notin E(G_1(A))$, which follows immediately from the definition of the edge set of $G_1(A)$:

$a_\alpha a_\beta \in E(G_1(A))$ if and only if $a_\beta \notin \cup \mathcal{F}_{a_\alpha}$.

As a result, the faces of a Type I matrix graph are independent subsets of $V(G_1(A))$. ■
Corollary 2 The normals of a Type I matrix graph are independent.

Proposition 7 Let $G_1 (A)$ be a Type I matrix graph, where $A \in \mathcal{M}^2_n$. Then $\alpha (G_1 (A)) = n$.

Proof. Note that in two dimensions, the notions of faces and normal are equivalent. Now, let $I$ be an arbitrary independent subset of $V (G_1 (A))$. Since $n \geq 2$, we can assume that $|I| \geq 2$. Moreover, by Proposition 6, each face of $A$ is an independent subset of $V (G_1 (A))$. It suffices to prove $I \subseteq F$ for some face $F$. To this end, suppose that $a_\alpha$ and $a_\beta$ are distinct vertices in $I$ with $\alpha = (i_1, j_1)$ and $\beta = (i_2, j_2)$. Observe that either $i_1 = i_2$ or $j_1 = j_2$, but not both. This is because otherwise, $a_\alpha a_\beta \in E (G_1 (A))$. Without loss of generality, we may assume that $\{a_{ij_1}, a_{ij_2}\} \subseteq I$ and that $j_1 \neq j_2$. Now consider an arbitrary $a_{kj} \in I$ and suppose to the contrary that $k \neq i$. Now, either $j \neq j_1$ or $j \neq j_2$. Consequently, either $a_{kj}a_{ij_1} \in E (G_1 (A))$ or $a_{kj}a_{ij_2} \in E (G_1 (A))$, which contradicts that fact that $a_{kj} \in I$. It follows that $I \subseteq F (a_\alpha) = \{a_{i_1}, a_{i_2}, \ldots, a_n\}$. Therefore, $|I| \leq n$ and $\alpha (G_1 (A)) = n$. ■

Although there will be no direct appeal to the next proposition, its proof might shed some light on a generalization of Theorem 4 in Section 4 below.

Proposition 8 Let $G_1 (A)$ be a Type I matrix graph, where $A \in \mathcal{M}^3_n$. Then $\alpha (G_1 (A)) = n^2$.

Proof. Let $I$ be an arbitrary independent subset of $V (G_1 (A))$. Since $n \geq 3$, we can assume that $|I| \geq 2$. Suppose now that $a_\alpha, a_\beta \in I$. Because $a_\beta \in \cup F_{a_\alpha}$, we can further assume, without loss of generality, that $a_\beta \in F_{i_3} (a_\alpha)$, for some direction $i_1 \in I_3$. First, the characteristics of a vertex $a_\gamma \in I \setminus F_{i_3} (a_\alpha)$ are established. There are two cases to consider.

Case 1 The vertices $a_\alpha$ and $a_\beta$ are contained in the same normal.

In this case, it will be demonstrated that $a_\alpha, a_\beta$, and $a_\gamma$ are all contained in some face of $A$. There exists a direction $i_2 \in I_3 \setminus \{i_1\}$ such that

$$a_\beta \in F_{i_3} (a_\alpha) \cap \eta_{F_{i_2}}^{-1} (a_\alpha).$$

We assert that there must exist a face $F$ of $A$ such that

$$a_\gamma \in F \in \mathcal{F}_{a_\alpha} \cap \mathcal{F}_{a_\beta}.$$

In fact, it will be shown that $F = F_{i_3} (a_\alpha)$. To see this, observe that because $k = 3$, we have

$$\eta_{F_{i_2}}^{-1} (a_\alpha) = F_{i_3} (a_\alpha) \cap F_{i_3} (a_\alpha),$$

where $i_3$ is the only remaining direction; i.e., $\{i_3\} = I_3 \setminus \{i_1, i_2\}$. Next, observe from (1) above, the following three conditions hold:

(i) $F_{i_3} (a_\beta) = F_{i_3} (a_\alpha)$
(ii) $F_{i_3} (a_\beta) = F_{i_3} (a_\alpha)$
(iii) $\eta_{F_{i_2}} (a_\beta) = \eta_{F_{i_2}} (a_\alpha)$.

From (1) again, it is emphasized that $F_{i_2} (a_\alpha) \neq F_{i_2} (a_\beta)$. Because $k = 3$, the face sets at $a_\alpha$ and at $a_\beta$ are given by

$$\mathcal{F}_{a_\alpha} = \{F_{i_1} (a_\alpha), F_{i_2} (a_\alpha), F_{i_3} (a_\alpha)\}$$

and

$$\mathcal{F}_{a_\beta} = \{F_{i_1} (a_\beta), F_{i_2} (a_\beta), F_{i_3} (a_\beta)\}.$$

Hence, from (i) and (ii) in (2) above, it follows that

$$\mathcal{F}_{a_\alpha} \cap \mathcal{F}_{a_\beta} = \{F_{i_1} (a_\alpha), F_{i_3} (a_\alpha)\}.$$

Continuing, we find that $a_\gamma \in F_{i_3} (a_\alpha) = F_{i_3} (a_\beta)$ because $a_\gamma \in F \in \mathcal{F}_{a_\alpha} \cap \mathcal{F}_{a_\beta}$ and we have assumed that $a_\gamma \notin F_{i_1} (a_\alpha)$. Therefore, it follows that $a_\alpha, a_\beta,$ and $a_\gamma$ are all contained in a single face of $A$. ■
Case 2  The vertices $a_\alpha$ and $a_\beta$ are not contained in the same normal.

In this case, it will be demonstrated that $a_\alpha, a_\beta,$ and $a_\gamma$ are all contained in the union of a normal set of $A$. Note that we are still under the assumption that $a_\beta \in F_{i_1} (a_\alpha)$. From this assumption, we find that

(iv) $\beta_{i_1} = \alpha_{i_1}$ and $\beta_1 \neq \alpha_i$ for $i \neq i_1$, and
(v) $a_\alpha, a_\beta \in F_{i_1} (a_\alpha) = F_{i_1, \alpha_{i_1}} (A) = F_{i_1, \beta_{i_1}} (A)$.  

Now, define the set

$$N = \left[ \bigcup_{i_1} \eta^+ (a_\alpha) \right] \cap \left[ \bigcup_{i_2} \eta^+ (a_\beta) \right].$$

We remark the elements of $N$ have coordinates that agree with the coordinates of $a_\alpha$ in all but one position, and agree with the coordinates of $a_\beta$ in all but one position, but the positions (one position from $a_\alpha$, and one position from $a_\beta$) of disagreement are not necessarily the same one position. Next we establish the existence of a vertex $a_\mu$ such that

(vi) $a_\mu \in N$
(vii) $a_\gamma \in \eta^+_{F_{i_3}} (a_\mu)$.

Should both conditions of (4) hold, it would then be the case that $a_\alpha, a_\beta,$ and $a_\gamma$ are all contained in the union of a normal set of $A$, namely $\bigcup_{i_3} \eta^+ (a_\mu)$. Since any two vertices in the union of a normal set are contained in at least one face of $A$, it is clear that the union of every normal set is an independent subset of $V(G_1 (A))$. To prove the existence of such a vertex $a_\mu$, we proceed as follows. Set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$. Note that $I_3 = \{1, 2, 3\} = \{i_1, i_2, i_3\}$ but it may not be the case that $i_t = t$ for $t = 1, 2, 3$. Since $k = 3$ and $|N| = 2$, there are exactly two possibilities for $a_\mu$. They are determined by defining the coordinates of $a_\mu$. The first is given by

$$\mu_{i_1} = \alpha_{i_1} = \beta_{i_1}$$
$$\mu_{i_2} = \alpha_{i_2}$$
$$\mu_{i_3} = \beta_{i_3},$$

and the second is given by

$$\mu_{i_1} = \beta_{i_1} = \alpha_{i_1}$$
$$\mu_{i_2} = \beta_{i_2}$$
$$\mu_{i_3} = \alpha_{i_3}.$$

There is a permutation $\sigma$ of $I_3$ such that $\mu = (\mu_{\sigma(i_1)}, \mu_{\sigma(i_2)}, \mu_{\sigma(i_3)})$. From either (5) or (6), it follows immediately that $a_\mu \in F_{i_1} (a_\alpha)$. Moreover, observe that $\mu$ agrees with $\alpha$ in all but one position and that $\mu$ agrees with $\beta$ in all but one position. Hence, we find that $a_\mu \in \bigcup_{i_3} \eta^+ (a_\alpha) \cap \bigcup_{i_3} \eta^+ (a_\beta)$; i.e., $a_\mu \in N$. Thus, (vi) holds in (4).

Next, it is demonstrated that $a_\gamma \in \eta^+_{F_{i_3}} (a_\mu)$. It suffices to show that $\gamma$ agrees with $\mu$ in all but one position. In fact, $\gamma$ does not agree with $\mu$ in position $i_1$ and $\gamma$ agrees with $\mu$ in positions $i_2$ and $i_3$. To see this, observe that in either case for $a_\mu$, we have $a_\mu \in F_{i_1} (a_\alpha)$ so that $\mu_{i_1} = \alpha_{i_1}$. However, $a_\gamma \in I \setminus F_{i_1} (a_\alpha)$ so that $\gamma_{i_1} \neq \alpha_{i_1}$. This confirms that $\gamma$ does not agree with $\mu$ in position $i_1$. Next, it is shown that $\gamma$ agrees with $\mu$ in positions $i_2$ and $i_3$. This requires more effort. Since $a_\gamma \in I$, there are faces $F$ and $F'$ with

$$a_\gamma \in F \in F_{a_\alpha} = \{F_{i_1} (a_\alpha), F_{i_2} (a_\alpha), F_{i_3} (a_\alpha)\}$$

and

$$a_\gamma \in F' \in F_{a_\beta} = \{F_{i_1} (a_\beta), F_{i_2} (a_\beta), F_{i_3} (a_\beta)\}.$$
Recall that we also have $a_\gamma \in \Gamma \setminus F_{i_1} \left( a_\alpha \right) = \Gamma \setminus F_{i_1} \left( a_\beta \right)$. We assert that $a_\gamma$ is a member of both the symmetric difference $F_{i_2} \left( a_\alpha \right) \otimes F_{i_3} \left( a_\alpha \right)$ and the symmetric difference $F_{i_2} \left( a_\beta \right) \otimes F_{i_3} \left( a_\beta \right)$. To prove the assertion, we first suppose to the contrary that $a_\gamma \notin F_{i_2} \left( a_\alpha \right) \cap F_{i_3} \left( a_\alpha \right)$. Then $\gamma_{i_2} = \alpha_{i_2}$ and $\gamma_{i_3} = \alpha_{i_3}$ so that $a_\gamma \notin \eta_{F_{i_1}} \left( a_\alpha \right)$. Since $\gamma_{i_2} \neq \alpha_{i_2}$, it follows that $a_{i_2}a_\gamma \notin E \left( G_1 \left( A \right) \right)$ contrary to $I$ being an independent subset of $V \left( G_1 \left( A \right) \right)$. For, $\gamma$ agrees with $\alpha$ in all positions except position $i_1$ and consequently in no position does $\gamma$ agree with $\beta$, which follows from (iv) in (3) above. Therefore, $a_\gamma \notin F_{i_2} \left( a_\alpha \right) \cap F_{i_3} \left( a_\alpha \right)$ and similarly $a_\gamma \notin F_{i_2} \left( a_\beta \right) \cap F_{i_3} \left( a_\beta \right)$. The assertion is now proven. Next, we suppose that $a_\gamma \in F_{i_2} \left( a_\alpha \right)$. Then $\gamma_{i_2} = \alpha_{i_2}$. From this, it cannot be that $a_\gamma \notin F_{i_2} \left( a_\beta \right)$ because this would imply that $\gamma_{i_2} = \beta_{i_2}$ and hence $\alpha_{i_2} = \beta_{i_2}$, which is a contradiction of (iv) in (3) above. Thus, it must be the case that $a_\gamma \in F_{i_3} \left( a_\beta \right)$ so that $\gamma_{i_3} = \beta_{i_3}$. It follows that

$$
\begin{align}
\gamma_{i_1} &\neq \alpha_{i_1} \text{ and } \gamma_{i_2} \neq \beta_{i_2} \\
\gamma_{i_2} &\neq \alpha_{i_2} \\
\gamma_{i_3} &\neq \beta_{i_3}
\end{align}
$$

The conditions in (5) and (6) the three conditions in (7) imply that $a_\gamma \in \eta_{F_{i_1}} \left( a_\mu \right)$ so that (vii) holds in (4) above. Therefore, $a_\alpha, a_\beta, a_\gamma \in \cup \eta_{i_{\alpha}} \left( a_\mu \right)$. \(\square\)

Now, since $n \geq k = 3$, it follows that

$$
k \left( n - 1 \right) + 1 \leq n \left( n - 1 \right) + 1 = n^2 - n + 1 < n^2.
$$

Moreover, $k \left( n - 1 \right) + 1$ is precisely the number of vertices contained in the union of a normal set by Proposition 2. From this fact, it can assumed that if $I$ is an independent set that contains more than $k \left( n - 1 \right) + 1$ vertices, then there exist two vertices of $I$ that are not contained in the same normal. Hence, if the vertices of $I$ are not all in the same face, it would follow by Case II that the independent set $I$ would be contained in the union of a normal set contradicting the fact that $I$ has more than $k \left( n - 1 \right) + 1$ vertices. Therefore, $I$ is contained in some face so that $|I| \leq n^2$. We conclude that $\alpha \left( G_1 \left( A \right) \right) = n^2$ since the faces of a Type I matrix graph are independent subsets having $n^2$ vertices. \(\blacksquare\)

**Corollary 3** Let $k \leq 3$. The faces of a Type I matrix graph determine independent subsets of maximum cardinality.

**Theorem 1** Let $G_1 \left( A \right)$ be a Type I matrix graph, where $A \in \mathcal{M}^k_n$. Then $\alpha \left( G_1 \left( A \right) \right) = n^{k-1}$.

**Proof.** The proof proceeds by induction on $k$. By Proposition 7 and Proposition 8, the result holds for $k = 2$ and $k = 3$. Inductively assume the result holds for $k = p$, where $p \geq 3$. It is shown that the result holds for $k = p + 1$. Let $A \in \mathcal{M}^{k+1}_n$, where $n \geq k + 1$ and consider a Type I matrix graph $G_1 \left( A \right)$ and an arbitrary independent subset $I$ of $V \left( G_1 \left( A \right) \right)$. It must be shown that $|I| \leq n^k$. By Corollary 1, $V \left( G_1 \left( A \right) \right)$ can be partitioned into $n$ subsets, each subset of which determines an induced subgraph of $G_1 \left( A \right)$ that is isomorphic to a Type I matrix graph $H_1 \left( C \right)$, where $C \in \mathcal{M}_n^k$. Call such a partition $\mathcal{P} = \left\{ P_1, P_2, \ldots, P_n \right\}$ so that $G_1 \left( A \right) \left[ P_i \right] \cong H_1 \left( C \right)$ for each $i \in I_n$. Observe that

$$
I = \bigcup_{i=1}^n \left( I \cap P_i \right).
$$

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By the inductive hypothesis, \( \alpha(H_1(C)) = n^{k-1} \) so that \( |I \cap P_i| \leq n^{k-1} \) for each \( i \in I_n \). Therefore,

\[
|I| = \sum_{i=1}^{n} |I \cap P_i| \leq \sum_{i=1}^{n} n^{k-1} = n^k.
\]

Because the faces of \( G_1(A) \) are independent subsets having \( n^k \) elements, it follows that

\[
\alpha(G_1(A)) = n^k.
\]

This completes the proof. ■

**Corollary 4** The faces of a Type I matrix graph determine independent subsets of maximum cardinality.

We are now in the position to determine the chromatic number of a Type I matrix graph.

**Lemma 1** Let \( G_1(A) \) be a Type I matrix graph, where \( A \in M_n^k \). Then \( G_1(A)[\Delta] \cong K_n \), where \( \Delta \) is the main diagonal of \( A \).

**Proof.** Let \( a_\alpha, a_\beta \in \Delta \). Then \( \alpha = c(1, 1, \ldots, 1) \) and \( \beta = d(1, 1, \ldots, 1) \) for some \( c, d \in I_n \) with \( c \neq d \). Therefore, it is immediately clear that

\[
a_\beta \notin \cup F_\alpha,
\]

since there does not exist a position in which \( \alpha \) and \( \beta \) agree. Because \( |\Delta| = n \), it follows that the subgraph of \( G_1(A) \) induced by \( \Delta \) is isomorphic to \( K_n \). ■

**Theorem 2** Let \( A \in M_n^k \). The Type I matrix graph \( G_1(A) \) is \( n \)-chromatic.

**Proof.** By Lemma 1, \( \chi(G_1(A)) \geq n \). Now, the faces of \( A \) are independent subsets of \( A^* \). Therefore, an \( n \)-coloring of \( G_1(A) \) can be exhibited by coloring each face \( F_i,j \) with color \( c_j \) for \( j = 1, 2, \ldots, n \), where \( i_0 \in I_k \) is a fixed direction. It follows that \( \chi(G_1(A)) = n \). ■

**Remark 2** Theorem 2 can also be obtained as a result of Proposition 1. For,

\[
\chi(G_1(A)) \geq \frac{|G_1(A)|}{\alpha(G_1(A))} = \frac{n^k}{n^{k-1}} = n
\]

and by coloring the faces as in the proof of Theorem 2, the desired result is obtained.

**Proposition 9** Let \( X = \{a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_t}\} \) be a set of \( t \) vertices in a Type I matrix graph \( G_1(A) \), where \( A \in M_n^k \) and \( 1 \leq t \leq n \). For \( i = 1, 2, \ldots, k \), define \( Z_i = \{\alpha_{i,j} : j \in I_i\} \). If \( |Z_i| = t \) for all \( i \in I_k \), then \( G_1(A)[Z_i] \cong K_t \).

**Proof.** The condition \( |Z_i| = t \) for all \( i \in I_k \) implies that there do not exist indices \( j_1, j_2 \in I_i \) and an index \( i_0 \in I_k \) for which \( \alpha_{i_0}^{j_1} = \alpha_{i_0}^{j_2} \). Hence,

\[
a_{\alpha_{j_1}} \notin \cup F_{a_{\alpha_{j_2}}}.
\]

Therefore, \( a_{\alpha_{j_1}} a_{\alpha_{j_2}} \in E(G_1(A)) \) and it follows that \( G_1(A)[Z_i] \cong K_t \). ■
4 On a generalization of completely independent critical cliques

The notation and terminology contained in this section are not standard. Therefore, some additional definitions are required. Recall that a vertex $v \in V(G)$ is a critical vertex of $G$ provided that the chromatic number of $G$ decreases upon the removal of $v$. In fact, the chromatic number decreases by exactly one whenever $v$ is a critical vertex; i.e., $\chi(G - v) = \chi(G) - 1$. Observe that the induced subgraph $G[{\{v}\}]$ satisfies $G[{\{v}\}] \cong K_1$. There is a natural generalization of this concept.

**Definition 13** Let $K$ be an $r$-clique of $G$. Then $K$ is a critical $r$-clique of $G$, written $K^c_r$, provided that $\chi(G - K) = \chi(G) - r$.

It is straightforward to prove any subgraph $K$ of order $r$ that satisfies the equation $\chi(G - K) = \chi(G) - r$ is necessarily isomorphic to $K^c_r$. Recall from Section 1 that a set $U$ of vertices is independent provided that no two vertices in $U$ are adjacent. Equivalently, $U$ is an independent subset of $V(G)$ whenever the induced subgraph $G[{U}]$ is isomorphic to an empty graph. The definition below of completely independent critical cliques can be viewed as a generalization of an independent set of vertices provided we consider each vertex in an independent set $U$ as an induced subgraph of $G$ isomorphic to $K_1$. However, we shall not adopt this point of view.

**Definition 14** Let $K^c_r$ and $K^c_s$ be two critical cliques of order $r$ and $s$, respectively. Then $K^c_r$ and $K^c_s$ are completely independent provided $N(v) \cap V(K^c_r) = \emptyset$ for every vertex $v \in V(K^c_s)$.

In Definition 14, $K^c_r$ and $K^c_s$ are completely independent provided that $\chi(G - K^c_r) = \chi(G) - r$ and $\chi(G - K^c_s) = \chi(G) - s$, and, moreover, no vertex in $K^c_r$ is adjacent to a vertex in $K^c_s$. The motivation for this definition arises out of its connection with a conjecture of Lovász in [6] that the only vertex double-critical graph is the complete graph. The double-critical conjecture has been proven in the affirmative by Stiebitz in [14] only in the case of a 5-chromatic double-critical graph. A more general statement that includes the conjecture of Lovász as a special case is the Erdős-Lovász Tihany conjecture. This more general conjecture and a brief history of some of the known results can be found in [8]. Related results for quasi-line graphs are given in [1]. The edge analogue of this conjecture has been resolved in the affirmative and can be found in [9] and [13]. It seems reasonable to believe that for a single critical clique $K^c_r$ there would be many edges from $K^c_r$ to $G - K^c_r$. Thus it might seem just as reasonable to believe for a family of critical cliques, that there would be many edges from one critical clique to the other. The results of this paper declare this is not the case. If $\mathcal{K} = \{K^c_{r_\alpha} : \alpha \in \Lambda\}$ is an indexed family of critical $r_\alpha$-cliques, then $\mathcal{K}$ is said to be a family of completely independent critical cliques provided that the elements of $\mathcal{K}$ are pairwise completely independent critical cliques. In [12], the existence of $\mathcal{K}$ for the case $|\mathcal{K}| = 2$ was addressed and it was demonstrated that there exists a vertex $k$-critical graph admitting two completely independent critical cliques having orders $r$ and $s$ for any $r$ and $s$, with $r, s \geq 1$. A 3-dimensional generalization of this notion is now given by establishing the existence of an infinite family of vertex critical graphs each admitting three completely independent critical cliques. This confirms the existence of $\mathcal{K}$ for $|\mathcal{K}| = 3$. 
4.1 The main construction

To begin the construction of such a family, select an arbitrary Type I matrix graph $G_1(A)$, where $A \in \mathcal{M}_3^3$. In what follows, the graph $G_1(A)$ will be referred to as the planet. Adjoin to the planet $G_1(A)$ three complete graphs of order $n - 1$, say

$$K_{n-1}^j(s_j) = \{x_{\bar{x}_1(s_j)}, x_{\bar{x}_2(s_j)}, \ldots, x_{\bar{x}_{n-1}(s_j)}\},$$

for $j = 1, 2, 3$. Here, an arbitrary 3-submultiset (or 3-combination) of $I_3$ has been chosen, $\{s_1, s_2, s_3\}$, as well as the fixed permutation $\sigma = 12 \cdots n$ of $I_3$. These three complete graphs will be referred to as the satellites. The reason why such a peculiar notation for these satellites is needed will become clear below.

As above, the family of satellites will be written as $K = \{K_{n-1}^j(s_j) : j = 1, 2, 3\}$. Construct a graph $G$ by defining $V(G)$ to be the set

$$V(G) = [V(G_1(A))] \cup [\cup K]$$

and by defining $E(G)$ according to the following two prescriptions indicated below.

(I) For all $a_{\alpha}, a_{\beta} \in V(G_1(A))$,

$$a_{\alpha}a_{\beta} \in E(G) \text{ if and only if } a_{\beta} \notin \cup F_{a_{\alpha}}.$$  

(II) For all $x^{j,t}_{1} \in \cup K$ and $a_{\alpha} \in V(G_1(A))$,

$$x^{j,t}_{1}a_{\alpha} \in E(G) \text{ if and only if } a_{\alpha} \notin \cup F_{x^{j,t}_{1}}.$$  

(6)

**Example 3** As an example to illustrate this labelling scheme, suppose that $\{1, 3, 4\}$ is chosen as the 3-submultiset of $I_4$. Also, let $\sigma = 1234$ so that for instance $\bar{\sigma}(2) = 134$. Further, we have $\bar{\sigma}_1(2) = 1$, $\bar{\sigma}_2(2) = 3$, and $\bar{\sigma}_3(2) = 4$. In this case, the vertices of the three satellites would be labelled as follows:

$$K_3^1(1) = \{x^{1,1}_2, x^{1,2}_3, x^{1,3}_4\},$$

$$K_3^2(3) = \{x^{2,1}_1, x^{2,2}_2, x^{2,3}_4\},$$

and

$$K_3^3(4) = \{x^{3,1}_1, x^{3,2}_2, x^{3,3}_3\}.$$

**Remark 3** In (II) of (6), it is somewhat cumbersome to visualize how vertices in the planet are adjacent to vertices in the satellites. An equivalent formulation of the second prescription is the following:

$$x^{j,t}_{k}a_{\alpha} \in E(G) \text{ if and only if } a_{\alpha} \in A^* \setminus \left(\bigcup_{i \neq j} F_{ik}(A)\right).$$

In other words, connect the vertex $x^{j,t}_{k}$ from the satellite $K_{n-1}^j(s_j)$ to all vertices that remain in the planet $G_1(A)$ after removing the $k$th face of $A$ in the $i$th direction for all directions except the $j$th direction. Figures 11-13 should make this more clear.
Figure 11: Vertices adjacent to $x_1^{1,1}$.

Figure 12: Vertices adjacent to $x_3^{2,1}$. 
In order to underscore the dependence of $\Gamma$ on the dimensions associated with the underlying $k$-dimensional $n$-square matrix graph and the set of satellites, we shall write $G = G_1^{k,n}(A, \mathcal{K})$. Here, $A$ denotes the associated $k$-dimensional $n$-square matrix and $\mathcal{K}$ represents the set of satellites, each member of which is a complete graph of order $n - 1$, adjoined to the planet $G_1(A)$. Also, observe now that the choice for the notation in the labelling of the vertices of each satellite is necessary as it provides a convenient method for describing exactly how the satellites are attached to the planet. The first objective is to determine $\chi \left( G_1^{3,n}(A, \mathcal{K}) \right)$.

**Theorem 3** The graph $G_1^{3,n}(A, \mathcal{K})$ is $(2n - 1)$-chromatic for every $n \geq 3$.

**Proof.** From Theorem 2 above, it follows that $\chi \left( G_1(A) \right) = n$. Clearly, $\chi \left( K_{n-1}^2(s_j) \right) = n - 1$. When the satellites are adjoined to form the graph $G_1^{3,n}(A, \mathcal{K})$, we claim that $\chi \left( G_1^{3,n}(A, \mathcal{K}) \right) = n + (n - 1) = 2n - 1$.

The proof proceeds by induction on $n$. For the base case of $n = 3$, consider the graph $G_1^{3,3}(A, \mathcal{K})$. It must be shown that $\chi \left( G_1^{3,3}(A, \mathcal{K}) \right) = 5$. By the definition of $G_1^{3,3}(A, \mathcal{K})$, the planet $G_1(A)$ is a Type I matrix graph with $A \in \mathcal{M}_3^3$ and so $\chi \left( G_1(A) \right) = 3$. Because each satellite is a 2-clique and there are no edges among distinct satellites by the definition of the edge set $E \left( G_1^{3,3}(A, \mathcal{K}) \right)$ in (6), it is clear that the chromatic number of each satellite is 2 and consequently, $\chi \left( G_1^{3,3}(A, \mathcal{K}) \right) \leq 5$. It remains to show that $\chi \left( G_1^{3,3}(A, \mathcal{K}) \right) \geq 5$. To this end, consider an arbitrary partition $\mathcal{P}$ of $V \left( G_1^{3,3}(A, \mathcal{K}) \right)$ into a minimal number of independent subsets. Every $X \in \mathcal{P}$ has the form $X = X_S \cup X_P$, where $X_S$ is an independent subset of $\cup \mathcal{K}$, the set of all satellite vertices; and, $X_P$ is an independent subset of $A^*$, the set of all planet vertices. Up to automorphism, there are exactly three distinct patterns as to how the vertices in the satellites can be adjacent to vertices in the planet. They are distinguished by the repetition numbers of the associated
submultiset of \( I_3 \). The submultiset that determines the pattern is given in parentheses.

**Table 1:** Pattern I (\([3, 3, 3]\))

<table>
<thead>
<tr>
<th>( X_S )</th>
<th>( Y_S )</th>
<th>( Z_S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K^1_2 ) (3): ( x^1,1 )</td>
<td>( x^1,2 )</td>
<td></td>
</tr>
<tr>
<td>( K^2_2 ) (3): ( x^2,1 )</td>
<td>( x^2,2 )</td>
<td></td>
</tr>
<tr>
<td>( K^3_2 ) (3): ( x^3,1 )</td>
<td>( x^3,2 )</td>
<td></td>
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</tbody>
</table>

**Table 2:** Pattern II (\([3, 3, 2]\))

<table>
<thead>
<tr>
<th>( X_S )</th>
<th>( Y_S )</th>
<th>( Z_S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K^1_2 ) (3): ( x^1,1 )</td>
<td>( x^1,2 )</td>
<td></td>
</tr>
<tr>
<td>( K^2_2 ) (3): ( x^2,1 )</td>
<td>( x^2,2 )</td>
<td></td>
</tr>
<tr>
<td>( K^3_2 ) (2): ( x^3,1 )</td>
<td>( x^3,2 )</td>
<td></td>
</tr>
</tbody>
</table>

**Table 3:** Pattern III (\([3, 2, 1]\))

<table>
<thead>
<tr>
<th>( X_S )</th>
<th>( Y_S )</th>
<th>( Z_S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K^1_2 ) (3): ( x^1,1 )</td>
<td>( x^1,2 )</td>
<td></td>
</tr>
<tr>
<td>( K^2_2 ) (2): ( x^2,1 )</td>
<td>( x^2,2 )</td>
<td></td>
</tr>
<tr>
<td>( K^3_2 ) (1): ( x^3,1 )</td>
<td>( x^3,2 )</td>
<td></td>
</tr>
</tbody>
</table>

In each pattern, the vertices of the satellites are represented in the rows of the table and the vertices in the columns will be grouped in various ways to represent how satellite vertices are distributed among the independent subsets contained in \( P \).

Now for each of these three patterns, there are two ways in which vertices in the columns can be colored; i.e., distributed among independent subsets of \( P \). These ways are indicated in (A) and (B) below.

(A) For each column, all vertices in the column are in a single color class.

(B) There exists a column for which not all of the vertices in the column are contained in a single color class.
For instance, in Pattern II, if the first type of coloration as in (A) is illustrated, then we would have

\[
X = X_S \cup X_P = \left\{ x_1^{1,1}, x_1^{2,1}, x_1^{3,1} \right\} \cup X_P
\]

\[
Y = Y_S \cup Y_P = \left\{ x_2^{1,2}, x_2^{2,2} \right\} \cup Y_P
\]

\[
Z = Z_S \cup Z_P = \left\{ x_3^{3,2} \right\} \cup Z_P
\]
as three representative elements from \( \mathcal{P} \). Also, in Pattern I, if the second type of coloration as in (B) is illustrated, then we would have

\[
X = X_S \cup X_P = \left\{ x_1^{1,1} \right\} \cup X_P
\]

\[
X' = X'_S \cup X'_P = \left\{ x_1^{2,1}, x_1^{3,1} \right\} \cup X'_P
\]

\[
Y = Y_S \cup Y_P = \left\{ x_2^{1,2}, x_2^{2,2}, x_2^{3,2} \right\} \cup Y_P
\]
as three representative elements from \( \mathcal{P} \). Observe that all vertices in the first column of Pattern I are shared (distributed) among two distinct color classes in \( \mathcal{P} \).

Consider the first type of coloration given by (A) for each pattern separately. For Pattern I, the arbitrary partition \( \mathcal{P} \) contains elements \( X \) and \( Y \) where

\[
X = \left\{ x_1^{1,1}, x_1^{2,1}, x_1^{3,1} \right\} \cup X_P
\]

and

\[
Y = \left\{ x_2^{1,2}, x_2^{2,2}, x_2^{3,2} \right\} \cup Y_P.
\]

Now consider the graph \( G_1^{3,3}(A, K) - X - Y \). To prove there are at least three elements of \( \mathcal{P} \setminus \{ X, Y \} \), it suffices to prove that there exists a subgraph of \( G_1^{3,3}(A, K) - X - Y \) that is isomorphic to \( K_3 \). To see that this is indeed the case, consider the set \( W = \{ a_{123}, a_{231}, a_{312} \} \). By Proposition 9, it follows that \( G_1^{3,3}(A, K)[W] \cong K_3 \). Moreover, \( W \subseteq V\left( G_1^{3,3}(A, K) - X - Y \right) \) since by the definition of the edge set of \( G_1^{3,3}(A, K) \), it must be that

\[
x_1^{1,1}a_{123}, x_1^{3,1}a_{231}, x_1^{2,1}a_{312} \in E\left( G_1^{3,3}(A, K) \right)
\]

and

\[
x_2^{2,2}a_{123}, a_2^{1,2}a_{231}, a_2^{3,2}a_{312} \in E\left( G_1^{3,3}(A, K) \right).
\]

Therefore, there must be at least three elements of the partition \( \mathcal{P} \) that remain upon the removal of \( X \) and \( Y \) from \( \mathcal{P} \).

In the next case, consider Pattern II. Here, the arbitrary partition \( \mathcal{P} \) contains elements \( X \), \( Y \), and \( Z \), where

\[
X = \left\{ x_1^{1,1}, x_1^{2,1}, x_1^{3,1} \right\} \cup X_P,
\]

\[
Y = \left\{ x_2^{1,2}, x_2^{2,2} \right\} \cup Y_P,
\]

and

\[
Z = \left\{ x_3^{3,2} \right\} \cup Z_P.
\]
As above, consider the graph $G_{1}^{3,3} (A, \mathcal{K}) - X - Y$. Clearly, either $a_{132} \in V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y \right)$ or $a_{221} \in V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y \right)$. This is because

$$x_{1}^{1,1} a_{132}, x_{1}^{3,1} a_{221}, a_{132} a_{221} \in E \left( G_{1}^{3,n} (A, \mathcal{K}) \right).$$

In the event that $a_{132} \in V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y \right)$, the set $W_{1} = \{ a_{132}, a_{213}, a_{321} \}$ satisfies

$$W_{1} \subseteq V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y \right)$$

and

$$G_{1}^{3,3} (A, \mathcal{K}) [W_{1}] \cong K_{3}.$$

And in the event that $a_{221} \in V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y \right)$, set $W_{2} = \{ a_{221}, x_{3}^{3,2}, a_{123}, a_{231}, a_{313} \}$ satisfies

$$W_{2} \subseteq V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y \right)$$

and

$$G_{1}^{3,3} (A, \mathcal{K}) [W_{2}] \cong C_{5},$$

the cycle on five vertices. In either case, there must exist at least three elements of $\mathcal{P}$ that remain upon the removal of $X$ and $Y$ from $\mathcal{P}$.

In the last case, consider Pattern III. The arbitrary partition $\mathcal{P}$ contains elements $X$, $Y$, and $Z$, where

$$X = \left\{ x_{1}^{1,1}, x_{1}^{2,1} \right\} \cup X_{P},$$

$$Y = \left\{ x_{2}^{1,2}, x_{2}^{3,1} \right\} \cup Y_{P},$$

and

$$Z = \left\{ x_{2}^{2,2}, x_{3}^{3,2} \right\} \cup Z_{P}.$$ 

Now, if these three elements of $\mathcal{P}$ are removed, then the set $W_{3} = \{ a_{132}, a_{213} \}$ satisfies

$$W_{3} \subseteq V \left( G_{1}^{3,3} (A, \mathcal{K}) - X - Y - Z \right)$$

and

$$G_{1}^{3,3} (A, \mathcal{K}) [W_{3}] \cong K_{2},$$

implying that there are at least five members of $\mathcal{P}$. Therefore, if the type of coloration in (A) is considered, it follows that $\chi \left( G_{1}^{3,3} (A, \mathcal{K}) \right) \geq 5$. Consequently, in this case, $\chi \left( G_{1}^{3,3} (A, \mathcal{K}) \right) = 5$.

Suppose now that the latter type of coloration in (B) is implemented. Then it is no longer necessary to consider separately the three patterns of adjacencies described above. Assume there exists a column in which there are two vertices in distinct satellites that are in distinct color classes. Call these vertices $x_{k}^{j_{1},t_{1}}$ and $x_{k}^{j_{2},t_{2}}$ and suppose they belong to the color classes $X$ and $Y$, respectively. Upon the removal of the color classes $X$ and $Y$ from the partition $\mathcal{P}$, there is a subgraph of $G_{1}^{3,3} (A, \mathcal{K}) - X - Y$, call it $H$, such that
$H$ is isomorphic to a graph of the form $G_{1}^{3,2} (B, K')$, where $B \in \mathcal{M}_{2}^{3}$ and $K'$ is family of 1-cliques. Note that in general, the eight elements of $B$ will be the vertices in the set

$$Z = \left[A^* \setminus \bigcup_{i \neq j_1} F_{ik} (A) \right] \cap \left[A^* \setminus \bigcup_{i \neq j_2} F_{ik} (A) \right].$$

The six edges between the satellites and planet of $G_{1}^{3,2} (B, K')$ can be selected from the up to eight remaining edges between the vertices that remain in the satellites and the planet of $G_{1}^{3,3} (A, K)$ when $x_{k}^{j_1, t_1}$, $x_{k}^{j_2, t_2}$, and $x_{k}^{j_3, t_0}$ are removed from the set of satellite vertices of $\cup K$, where $j_3 \in I_3 \setminus \{j_1, j_2\}$ and for some $t_0 \in \{1, 2\}$. However, the edges that remain between the satellites and the planets do not play a role in this portion of the proof. The only thing that matters is how the eight vertices in $Z$ are distributed among the color classes contained in $P$. Furthermore, although the graph $G_{1}^{3,2} (B, K')$ will not in general be part of the family of interest since $2 < 3$, it is constructed in the same fashion as when $n \geq k$. This is the only time that a graph of the form $G_{1}^{3,2} (B, K')$ will be considered in this paper.

![Figure 14: The planet $G_{1} (B)$ of $H$.](image_url)

There are exactly three colorations, up to automorphism, of the planet $G_{1} (B)$ shown in Figure 14 above. The light gray lines are for visual purposes only. However, it suffices to consider only two types of colorations. These ways are indicated in (C) and (D) as follows:

(C) Three colors are used to color $G_{1} (B)$
(D) Two colors are used to color $G_{1} (B)$.

In the event that three colors have been used to color $G_{1} (B)$ as in (C), it is then clear that at least five colors have been used in the coloration of $G_{1}^{3,3} (A, K)$. On the other hand, in the event that only two colors have been used to color the graph $G_{1} (B)$ as in (D), we assert that five colors must be used to color just the planet $G_{1} (A)$ itself. To prove this assertion, it suffices to exhibit three vertices of $G_{1} (A)$ such that the subgraph of $G_{1} (A)$ induced by this set of three vertices is isomorphic to $K_{5}$; and, that each of these three vertices is adjacent to at least one vertex from each color class in the two coloring of $G_{1} (B)$ being considered.
in (D). In the Figure 15 below, we have embedded an isomorphic copy of $G_1 (B)$ in the graph $G_1 (A)$. The light gray lines are for visual purposes only and the gray lines demonstrate a subgraph isomorphic to $K_3$ induced by the set of vertices that colored green, yellow, and coral.

![Figure 15: Vertices of an induced $K_3$.](image)

The three vertices are selected as follows. Choose a representative 3-matching normal basis such that each normal contains two differently colored vertices from the embedded 3-dimensional 2-square matrix graph. There will be one vertex remaining in each normal. These three vertices are the ones sought. (What we mean is this: Consider the light gray lines in the embedded planet $G_1 (B)$. Choose a set of three normals in $G_1 (B)$ such that the two vertices in each normal are colored differently and that the three normals form a 3-matching in the planet $G_1 (B)$. Now consider the extension of these three normals to three normals in the planet $G_1 (A)$. Select one vertex from each of these three normals in $G_1 (A)$ that was not a vertex in the corresponding normal in $G_1 (B)$.) By Proposition 9, these three vertices determine an induced subgraph isomorphic to $K_3$. This proves that \( \chi \left( G_{11}^{3,3} (A, K) \right) = 5 \) and establishes the base case.

Inductively assume that \( \chi \left( G_{11}^{3,r} (A, K) \right) = 2r - 1 \) and consider the graph $G_{1}^{3,r+1} (A, K)$. Let $\mathcal{P}$ be an arbitrary partition of $G_{1}^{3,r+1} (A, K)$ into a minimal number of independent subsets. As in the base case, there are exactly three ways, up to automorphism, that vertices in the satellites can be adjacent to vertices in the planet. These patterns are indicated below.
Table 4: Pattern I ({3, 3, 3})

<table>
<thead>
<tr>
<th></th>
<th>$X_{S,1}$</th>
<th>$X_{S,2}$</th>
<th>$X_{S,3}$</th>
<th>$X_{S,4}$</th>
<th>$X_{S,r}$</th>
<th>$X_{S,r+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_2^1$ (3)</td>
<td>$x_1^{1,1}$</td>
<td>$x_2^{1,2}$</td>
<td>$x_4^{1,3}$</td>
<td>$x_r^{1,r-1}$</td>
<td>$x_r^{1,r}$</td>
<td>$x_{r+1}$</td>
</tr>
<tr>
<td>$K_2^2$ (3)</td>
<td>$x_1^{2,1}$</td>
<td>$x_2^{2,2}$</td>
<td>$x_4^{2,3}$</td>
<td>$x_r^{2,r-1}$</td>
<td>$x_r^{2,r}$</td>
<td>$x_{r+1}$</td>
</tr>
<tr>
<td>$K_2^3$ (3)</td>
<td>$x_1^{3,1}$</td>
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<td>$x_r^{3,r-1}$</td>
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Table 5: Pattern II ({3, 3, 2})

<table>
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<tr>
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<th>$X_{S,1}$</th>
<th>$X_{S,2}$</th>
<th>$X_{S,3}$</th>
<th>$X_{S,4}$</th>
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</tr>
<tr>
<td>$K_2^2$ (3)</td>
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<td>$x_4^{2,3}$</td>
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</tr>
<tr>
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<td>$x_3^{3,2}$</td>
<td>$x_4^{3,3}$</td>
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<td>$x_r^{3,r}$</td>
<td>$x_{r+1}$</td>
</tr>
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</table>

Table 6: Pattern III ({3, 2, 1})

<table>
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<th>$X_{S,1}$</th>
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<th>$X_{S,3}$</th>
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<th>$X_{S,r}$</th>
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<tbody>
<tr>
<td>$K_2^1$ (3)</td>
<td>$x_1^{1,1}$</td>
<td>$x_2^{1,2}$</td>
<td>$x_4^{1,3}$</td>
<td>$x_r^{1,r-1}$</td>
<td>$x_r^{1,r}$</td>
<td>$x_{r+1}$</td>
</tr>
<tr>
<td>$K_2^2$ (2)</td>
<td>$x_1^{2,1}$</td>
<td>$x_3^{2,2}$</td>
<td>$x_4^{2,3}$</td>
<td>$x_r^{2,r}$</td>
<td>$x_r^{2,r}$</td>
<td>$x_{r+1}$</td>
</tr>
<tr>
<td>$K_2^3$ (1)</td>
<td>$x_2^{3,1}$</td>
<td>$x_3^{3,2}$</td>
<td>$x_4^{3,3}$</td>
<td>$x_r^{3,r-1}$</td>
<td>$x_r^{3,r}$</td>
<td>$x_{r+1}$</td>
</tr>
</tbody>
</table>
Now for each of these three patterns, there are two ways in which vertices in the columns can be colored; i.e., distributed among independent subsets of \( V \left( G^3_1 (A, K) \right) \). These ways are exactly as they were in the base case above. The are restated for convenience.

A. For each column, all vertices in the column are in a single color class.

B. There exists a column for which not all of the vertices in the column are contained in a single color class.

Let us assume the former coloration in (A) is implemented. For Pattern I, after \( \rho \) elements of the partition \( P \) are removed, the vertices in the set \( \Pi = \alpha(1, 2, 3) \cup \alpha(2, 3, 4) \cup \alpha(3, 4, 5) \cup \cdots \cup \alpha(\rho, \rho + 1, 1) \cup \alpha(\rho + 1, 1, 2) \) determine, by Proposition 9, an induced subgraph \( \Gamma[\Pi] \) satisfying the isomorphism \( \Gamma[\Pi] \cong K_{r+1} \). The fact that these vertices do in fact remain, is confirmed by the chart of adjacencies below. In the chart, a planet vertex appearing in column \( p \) is adjacent to every satellite vertex appearing in column \( q \).

\[ \begin{array}{c|cccc}
\Pi_{S, 1} & a(1, 2, 3) & a(2, 3, 4) & \cdots & a(r, r+1, 1) & a(r+1, 1, 2) \\
\hline
X_{S, 1} & x_{1,1} & x_{1,1} & \cdots & x_{1,1} & x_{1,1} \\
& x_{1,2} & x_{1,2} & \cdots & x_{1,2} & x_{1,2} \\
X_{S, 2} & x_{2,2} & x_{2,2} & \cdots & x_{2,2} & x_{2,2} \\
& x_{2,3} & x_{2,3} & \cdots & x_{2,3} & x_{2,3} \\
X_{S, 3} & x_{3,3} & x_{3,3} & \cdots & x_{3,3} & x_{3,3} \\
& x_{3,4} & x_{3,4} & \cdots & x_{3,4} & x_{3,4} \\
X_{S, r} & x_{r,r-1} & x_{r,r-1} & \cdots & x_{r,r-1} & x_{r,r-1} \\
& x_{r,r} & x_{r,r} & \cdots & x_{r,r} & x_{r,r} \\
X_{S, r+1} & x_{r+1,r} & x_{r+1,r} & \cdots & x_{r+1,r} & x_{r+1,r} \\
\end{array} \]

Therefore, we see that there must be at least \( r + 1 \) elements of \( P \) that remain upon the removal of the color \( r \) classes \( X_{S, 1}, X_{S, 2}, X_{S, 4}, X_{S, 5}, \ldots, X_{S, r}, X_{S, r+1} \).

For Pattern II, after removing up to \( r + 1 \) elements from the partition \( P \), we have three subcases to consider.
Case 1 (Coloration A, Pattern II) The equation \( r + 1 = 4 \) is true so that \( r = 3 \).
In this case, the set
\[
X = \{a_{124}, a_{241}, a_{413}\}
\]
satisfies \( G_1^{3, r+1} (A, K) [X] \cong K_3 \) by Proposition 9 so that
\[
\chi \left( G_1^{3, r+1} (A, K) \right) = 4 + 3 = 7 = 2 (r + 1) - 1.
\]

Case 2 (Coloration A, Pattern II) The equation \( r + 1 = 5 \) is true so that \( r = 4 \).
In this case, the set
\[
X = \{a_{123}, a_{245}, a_{541}, a_{514}\}
\]
satisfies \( G_1^{3, r+1} (A, K) [X] \cong K_4 \) by Proposition 9 so that
\[
\chi \left( G_1^{3, r+1} (A, K) \right) = 5 + 4 = 9 = 2 (r + 1) - 1.
\]

Case 3 (Coloration A, Pattern II) The inequality \( r + 1 > 5 \) is true so that \( r > 4 \).
In the last case, after removing the \( r - 2 \) elements \( X_4, X_5, \ldots, X_r, X_{r+1} \), the set
\[
X = \{a_{(4,5,6)}, a_{(5,6,7)}, \ldots, a_{(r-1, r, r+1)}, a_{(r, r+1, 4)}, a_{(r+1, 4, 5)}\}
\]
satisfies \( G_1^{3, r+1} (A, K) [X] \cong K_{r-2} \) by Proposition 9 so that, in addition to the subgraph \( H \) defined above which requires 5 colors, we have
\[
\chi \left( G_1^{3, r+1} (A, K) \right) = (r - 2) + (r - 2) + 5 = 2 (r + 1) - 1.
\]

For Pattern III, observe that if \( r + 1 > 5 \), then this pattern would be the same as Pattern II. Hence, we may assume that \( r + 1 \leq 5 \). There are two cases to consider after removing all \( r + 1 \) partition elements.

Case 1 (Coloration A, Pattern III) The equation \( r + 1 = 4 \) is true so that \( r = 3 \).
In this case, the set
\[
X = \{a_{134}, a_{243}, a_{412}\}
\]
satisfies \( G_1^{3, r+1} (A, K) [X] \cong K_3 \) by Proposition 9 so that
\[
\chi \left( G_1^{3, r+1} (A, K) \right) = 4 + 3 = 7 = 2 (r + 1) - 1.
\]

Case 2 (Coloration A, Pattern III) The equation \( r + 1 = 5 \) is true so that \( r = 4 \).
In this case, the set
\[
X = \{a_{134}, a_{245}, a_{452}, a_{513}\}
\]
satisfies \( G_1^{3, r+1} (A, K) [X] \cong K_4 \) by Proposition 9 so that
\[
\chi \left( G_1^{3, r+1} (A, K) \right) = 5 + 4 = 9 = 2 (r + 1) - 1.
\]

Suppose now that the latter type of coloration is implemented. In this event, it no longer becomes necessary to consider separately the three patterns of adjacencies described above. Assume that there exists a column in which there are two vertices in distinct satellites that are in distinct color classes. Call these vertices \( x_k^{j_1, t_1} \) and \( x_k^{j_2, t_2} \). Moreover, suppose these two vertices belong to the color classes \( X \) and \( Y \), respectively. Upon the removal of the color classes \( X \) and \( Y \) from the partition \( P \), we assert that there exists a subgraph \( H \) of \( G_1^{3, r+1} (A, K) \), in the remaining subgraph, that is isomorphic to \( G_1^{3, r} (A, K) \). To see this, we consider the following two cases.
Case 1 (Coloration B) When the color classes $X$ and $Y$ are removed from $P$, at most one vertex from each satellite is removed. In this case, $r - 1$ vertices remain in each clique when $X$ and $Y$ are removed. Therefore the is a subgraph $H$ of $G_1^{3, r+1} (A, \mathcal{K})$ that is isomorphic to $G_1^{3, r} (A, \mathcal{K})$.

Case 2 (Coloration B) When the color classes $X$ and $Y$ are removed from $P$, there exists a satellite for which two vertices have been removed. Note that there cannot be more than two vertices removed from any satellite when $X$ and $Y$ are removed from $P$. For each such satellite that have two vertices removed, there is a vertex that can be taken from the planet and moved into orbit to serve as a new satellite vertex. If this can be established for each satellite that had two vertices removed, then the previous case applies, in which each satellite would have $r - 1$ vertices. Suppose that $K^j_2 (s_j)$ is a satellite that had two vertices removed. Recall that

$$K^j_2 (s_j) = \{ x^{j, 1}_{\sigma_1 (s_j)}, x^{j, 2}_{\sigma_2 (s_j)}, \ldots, x^{j, r}_{\sigma_r (s_j)} \}.$$ 

Further assume that vertices $x^{j, n_1}_{\sigma_{n_1} (s_j)}$ and $x^{j, n_2}_{\sigma_{n_2} (s_j)}$ have been removed upon the removal of $X$ and $Y$. Define $\beta (j) \in I^n_\alpha$ by the rule

$$(\beta (j))_i = \begin{cases} j & \text{if } i = j \\ s_j & \text{if } i \neq j \end{cases}.$$ 

By the definitions of the satellite $K^j_2 (s_j)$ and the edge set $G_1^{3, n} (A, \mathcal{K})$, the vertex $a_\beta (j)$ is adjacent to every vertex in $K^j_2 (s_j)$. Therefore, $a_\beta (j)$ remains upon the removal of $X$ and $Y$. Thus, the new satellite

$$\{ a_\beta (j) \} \cup K^j_2 (s_j) \setminus (X \cup Y)$$

has order $r - 1$. We remark that it is not required that $a_\beta (j)$ be independent from the other satellites. This is because we are only interested in finding a subgraph $H$ that is isomorphic to $G_1^{3, r} (A, \mathcal{K})$.

Now, in the case of Coloration B, we have shown that in the subgraph $G_1^{3, r+1} (A, \mathcal{K}) - X - Y$, there exists a subgraph $H$ of satisfying $H \cong G_1^{3, r} (A, \mathcal{K})$. Therefore, by the inductive hypothesis,

$$\chi \left( G_1^{3, r+1} (A, \mathcal{K}) - (X \cup Y) \right) \geq \chi \left( G_1^{3, r} (A, \mathcal{K}) \right) = 2r - 1.$$ 

Consequently, $\chi \left( G_1^{3, r+1} (A, \mathcal{K}) \right) \geq (2r - 1) + 2 = 2r + 1$ and since one can easily exhibit a $(2r + 1)$-coloring of $G_1^{3, r+1} (A, \mathcal{K})$ by coloring the planet with $r + 1$ colors and coloring the satellites with an additional $r$ colors, it follows that $\chi \left( G_1^{3, r+1} (A, \mathcal{K}) \right) = 2r + 1$. By the principle of mathematical induction, we conclude that $\chi \left( G_1^{3, n} (A, \mathcal{K}) \right) = 2n - 1$ for all $n \geq 3$. ■
Theorem 4 The set $G = \{G_{1}^{3,n}(A, K) : n \geq 3\}$ is a family of graphs satisfying the following conditions.

1. $\chi(G_{1}^{3,n}(A, K)) = 2n - 1$.

2. The induced subgraphs $G_{1}^{3,n}(A, K)[K_{n-1}^{j}(s_{j})]$, for $j = 1, 2, 3$, are completely independent critical $(n - 1)$-cliques.

Proof. By Theorem 3, the graph $G_{1}^{3,n}(A, K)$ is $(2n - 1)$-chromatic. For any direction $i$, where $1 \leq i \leq 3$, color the $j$th face of $A$ in the $i$th direction with color $c_{j}$ for $j \in I_{n}$. Necessarily, the satellite $K_{n-1}^{i}(s_{i})$ would require $n - 1$ additional colors, say $c_{n+1}, c_{n+2}, \ldots, c_{2n-1}$, while the remaining satellites can be colored from among the colors $c_{1}, c_{2}, \ldots, c_{n}$. This coloration shows that the satellite $G_{1}^{3,n}(A, K)[K_{n-1}^{i}(s_{i})]$ is a critical $(n - 1)$-clique. By the definition of $E(G_{1}^{3,n}(A, K))$, these cliques are also completely independent.

Corollary 5 For every $p, q, r \in \mathbb{N}$, where $p, q, r \geq 2$, there exists a vertex critical graph which admits critical cliques having orders $p, q,$ and $r$ that are completely independent.

5 Concluding Remarks

It seems as though graphs admitting completely independent critical cliques are rare. It would be interesting to determine whether or not there are other families of graphs admitting completely independent critical cliques; or to be able to classify graphs which have this property. Also, what condition can be imposed on a graph to guarantee that a certain number of edges between critical cliques must exist? The answer to this question might lead to answering double-critical conjecture of Lovász in the affirmative. The generalization from $G_{1}^{3,n}(A, K)$ to $G_{k,n}(A, K)$ at present is incomplete but will be investigated further. We conclude by with the following conjecture.

Conjecture 1 For every pair $m, n \in \mathbb{N}$, there exists a vertex critical graph admitting $m$ completely independent critical cliques of order $n$.

Thus far, this conjecture holds for all $n$ and $m = 1, 2,$ and $3$. 
References


