RANDOM CHANNEL MODELING IN COMMUNICATION SYSTEMS ANALYSIS AND DESIGN

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ABSTRACT: A theory for modeling channels as random processes impulse response is presented and applied to the analysis of communication systems. Channels are characterized statistically via probability density functions of describing parameters that yield the channel mean and autocorrelation function as well as spectrally via the equivalent of a transfer function. Generalized channel input/output relationships are derived for random process input models. A generalization of the classical result applicable to deterministic channel models via an impulse response is obtained and applied to representative examples in communications engineering [1].

A communication channel is normally described in terms of its impulse response $h(t,\tau)$, namely its response at time t to an impulse applied at time τ. Generally such model is treated as a deterministic function of two variables. However in most practical applications uncertainties about certain elements characterizing the channel make the deterministic model unrealistic. While the general characteristics of the channel may be known (i.e., low pass, or bandpass with a certain rolloff) the uncertain elements associated with the channel (i.e., attenuation, cutoff, group delay) require a more realistic modeling in terms of random processes that take such uncertainties into account. Efforts at random channel modeling dates back to the work found in [2,3] and more recently in the work on design of robust filters [4]. Therefore a random channel is modeled as an ensemble of impulse responses $\{h_i(t,\tau)\}$ that form the random process $H(t,\tau)$ which in turns becomes the random channel model. The most complete model allows for the possibility that these realizations are time varying impulse responses while the random process itself is nonstationary. The complexity of analyzing a channel that is both time varying and (statistically) non-stationary exceeds the scope of this work. It will therefore be assumed that while the channel may be nonstationary, it is time invariant thereby resulting in a simplified model that allows for the channel to be described in terms of $H(t,0)$ which is equivalently described by the simplified notation $H(t)$. A random channel will be assumed to be sufficiently known so that basic statistical information about it can be obtained, such as the channel mean and the channel autocorrelation function. Ultimately the goal is to be able to obtain a statistical description of the channel output from knowledge of the input and the channel itself. With $X(t)$ and $Y(t)$ the channel input and output random processes respectively, the following notation will be used,

$E\{X(t)\} = m_X(t) =$ mean of the input random process (1)

$E\{X(t_1)X(t_2)\} = R_{XX}(t_1,t_2) =$ autocorrelation function of the input random process (2)

$E\{H(t)\} = m_H(t) =$ mean of the channel impulse response (3)

$E\{H(t_1)H(t_2)\} = R_{HH}(t_1,t_2) =$ autocorrelation function of the channel impulse response (4)

$E\{Y(t)\} = m_Y(t) =$ mean of the output random process (5)

$E\{Y(t_1)Y(t_2)\} = R_{YY}(t_1,t_2) =$ autocorrelation function of the output random process (6)

Basic system theory relates the channel output to its input and the impulse response, via the convolution relation,

$$Y(t) = \int_{-\infty}^{\infty} H(\xi)X(t-\xi)d\xi \tag{7}$$

Note that Eq. 7 is only symbolic and must be viewed in terms of individual realizations that collectively form the random processes that are part of the convolution relationship. Throughout this work it will be assumed that the input $X(t)$ and the channel $H(t)$ are statistically independent random processes. Additionally, the classical input-output system relationships for deterministic channels expressed in the frequency domain can be extended to the random channel model, especially for cases involving wide sense stationary (w.s.s.) random processes.

1. General Results on the Output Statistics

The statistics of the channel output process $Y(t)$ are developed first in their most general form and in terms of channel statistics specified by its mean and autocorrelation function. These results can be used to analyze specific cases involving channel models and assumed random process inputs in order to determine how the channel affects the input so as to produce an output that can also be characterized statistically in terms of its mean and autocorrelation function. From Eq. 7, the output mean is easily obtained from

$$E\{Y(t)\} = E\left\{\int_{-\infty}^{\infty} H(\xi)X(t-\xi)d\xi\right\} = \int_{-\infty}^{\infty} E\{H(\xi)X(t-\xi)\}d\xi = \int_{-\infty}^{\infty} E\{H(\xi)\}E\{X(t-\xi)\}d\xi \quad (1.1)$$

The simplification of the expectation operation inside the integral is made possible by the assumption that the channel and the input are statistically independent random processes. When the input process additionally is w.s.s. so that the expectation $E\{X(t-\xi)\}$ is constant, say m_X, then

$$E\{Y(t)\} = m_X \int_{-\infty}^{\infty} E\{H(\xi)\}d\xi = m_X \int_{-\infty}^{\infty} m_H(\xi)d\xi \quad (1.2)$$

where the notation of Eq. 3 has been used in Eq. 1.2. Observe that unless $m_X = 0$, Eq. 1.2 yields the result that a nonzero mean input can produce an output process that has an unbounded mean. This would definitely occur if a stationary (nonzero mean) model is applicable to the channel or the integral of Eq. 1.2 diverges. This result would tend to point to the observation that "reasonable" random channel models are nonstationary processes. Now,

$$\begin{aligned} E\{Y(t_1)Y(t_2)\} &= E\left\{\int_{-\infty}^{\infty} H(\xi)X(t_1-\xi)d\xi \int_{-\infty}^{\infty} H(\eta)X(t_2-\eta)d\eta\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{H(\xi)X(t_1-\xi)H(\eta)X(t_2-\eta)\}d\xi d\eta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{H(\xi)H(\eta)\}E\{X(t_1-\xi)X(t_2-\eta)\}d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)R_{XX}(t_1-\xi,t_2-\eta)d\xi d\eta = R_{YY}(t_1,t_2) \end{aligned} \quad (1.3)$$

When the input process is w.s.s., then $R_{XX}(t_1-\xi,t_2-\eta) = R_{XX}(t_1-t_2-(\xi-\eta))$ so that Eq. 1.3 becomes

$$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)R_{XX}(t_1-t_2-(\xi-\eta))d\xi d\eta = R_{YY}(t_1-t_2) \quad (1.4)$$

The results of Eqs. 1.2 and 1.4 clearly demonstrate that when the channel input is a w.s.s. random process, so is the channel output process. The fact that the channel may be nonstationary when modeled as a random process does not affect well-known results about stationarity of the output [1]. With $t_1 = t+\tau$ and $t_2 = t$, Eq. 1.3 becomes

$$E\{Y(t+\tau)Y(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)R_{XX}(t+\tau-\xi,t-\eta)d\xi d\eta = R_{YY}(t+\tau,t) \quad (1.5)$$

Applying to Eq. 1.5 the standard time averaging operation for which the notation $\langle \bullet \rangle = \lim_{T\to\infty} T^{-1}\int_{-T/2}^{T/2} \bullet dt$ is used,

$$\bar{R}_{YY}(\tau) = \langle R_{YY}(t+\tau,t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)\langle R_{XX}(t+\tau-\xi,t-\eta)\rangle d\xi d\eta \quad (1.6)$$

When the input process is w.s.s., time averages are not necessary, so that Eq. 1.5 becomes

$$E\{Y(t+\tau)Y(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)R_{XX}(\tau-(\xi-\eta))d\xi d\eta = R_{YY}(\tau) \quad (1.7)$$

Using $\Im\{\bullet\}$ to denote standard one dimensional or two-dimensional Fourier Transforms as appropriate, then from the Wiener-Khinchin theorem applicable to the w.s.s case, [2], $\Im\{R_{XX}(\tau)\} = S_{XX}(f)$ and $\Im\{R_{YY}(\tau)\} = S_{YY}(f)$ represent PSD's of the input and output processes, respectively, so that from Eq. 1.7

$$S_{YY}(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)\Im\{R_{YY}(\tau-(\xi-\eta))\}d\xi d\eta \quad (1.8)$$

Since $\Im\{R_{XX}(\tau-(\xi-\eta))\} = S_{XX}(f)e^{-j2\pi f(\xi-\eta)}$, obtain from Eq. 1.8

$$S_{YY}(f) = S_{XX}(f)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)e^{-j2\pi f(\xi-\eta)}d\xi d\eta = S_{XX}(f)S_{HH}^{d}(f,f) \tag{1.9}$$

where the 2-D Fourier Transform used to obtain the compact result of Eq. 1.9 is given by

$$S_{HH}^{d}(f,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{HH}(\xi,\eta)e^{-j2\pi(f\xi-\nu\eta)}d\xi d\eta \tag{1.10}$$

For the more general case of Eq. 1.6, since

$$\begin{aligned}\langle R_{XX}(t+\tau-\xi,t-\eta)\rangle &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} R_{XX}(t+\tau-\xi,t-\eta)dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2-\eta}^{T/2-\eta} R_{XX}(\lambda+\tau-(\xi-\eta),\lambda)d\lambda \\ &= \bar{R}_{XX}(\tau-(\xi-\eta))\end{aligned} \tag{1.11}$$

the more general case produces a result essentially identical to that of Eq. 1.9 except that $S_{XX}(f)$ is replaced by $\bar{S}_{XX}(f) = \Im\{\bar{R}_{XX}(\tau)\}$. Observe also in Eq. 1.9 that $S_{HH}^{d}(f,f)$ takes on the role of the magnitude squared of the channel transfer function in the well known case using a deterministic channel model. It is not difficult to show that regardless of channel model, as long as $R_{HH}(t_1,t_2)$ is real, $S_{XX}^{d}(f,f)$ is real, even, and nonnegative.

2. Application to a Baseband Channel Model

Consider a specific random channel model with ideal low pass characteristics described by the impulse response

` $$H(t) = 2AB\frac{\sin 2\pi Bt}{2\pi Bt} \tag{2.1}$$

where the gain and cutoff are specified by the random variables A and B, respectively. For such a channel,

$$E\{H(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2a\frac{\sin(2\pi bt)}{2\pi t} f_{AB}(a,b)dadb \tag{2.2}$$

and

$$E\{H(t_1)H(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2a\frac{\sin(2\pi bt_1)}{2\pi t_1}2a\frac{\sin(2\pi bt_2)}{2\pi t_2} f_{AB}(a,b)dadb \tag{2.3}$$

Evaluation of these two expectations ultimately requires knowledge of the joint pdf of the gain and cutoff channel characteristics. Channel characterization in the frequency domain in principle can be obtained from

$$S_{HH}^{d}(f,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2a\frac{\sin(2\pi bt_1)}{2\pi t_1}2a\frac{\sin(2\pi bt_2)}{2\pi t_2} f_{AB}(a,b)dadb e^{-j2\pi(ft_1-\nu t_2)}dt_1dt_2 \tag{2.4}$$

Carrying out the integration over the time variables first, and using the well-known Fourier Transform pair

$$2\frac{\sin 2\pi Wt}{2\pi t} \longleftrightarrow rect\left(\frac{f}{2W}\right) \quad ; \quad rect(\lambda) = \begin{cases}1 & |\lambda| \le 1/2 \\ 0 & otherwise\end{cases} \tag{2.5}$$

yields

$$S_{HH}^{d}(f,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a\,rect\left(\frac{f}{2b}\right) a\,rect\left(-\frac{\nu}{2b}\right) f_{AB}(a,b)dadb \tag{2.6}$$

As a more specific example, consider the case where the channel gain and cutoff are statistically independent and modeled as Rayleigh and Uniform random variables, respectively. Thus,

$$f_{AB}(a,b) = f_A(a) f_B(b) = \frac{a}{\sigma^2} e^{-a^2/2\sigma^2} u(a)\left[\frac{1}{b_2-b_1}[u(b-b_1)-u(b-b_2)]\right] \tag{2.7}$$

Obtain therefore from Eq. 2.2,

$$E\{H(t)\} = \int_{-\infty}^{\infty} 2a f_A(a) da \int_{-\infty}^{\infty} \frac{\sin(2\pi bt)}{2\pi t} f_B(b) db = 2E\{A\}\int_{b_1}^{b_2} \frac{1}{b_2-b_1}\frac{\sin(2\pi bt)}{2\pi t} db$$

$$= \frac{2E\{A\}}{(2\pi t)^2}\left[-\frac{\cos(2\pi bt)}{b_2-b_1}\right]_{b_1}^{b_2} = \sqrt{\frac{\pi}{2}}\sigma[\sin c(b_2-b_1)t]\sin[\pi(b_1+b_2)t] = m_H(t) \tag{2.8}$$

Similarly, from Eq. 2.3

$$E\{H(t_1)H(t_2)\} = \int_{-\infty}^{\infty} 4a^2 f_A(a) da \int_{-\infty}^{\infty} \frac{\sin(2\pi b t_1)}{2\pi t_1}\frac{\sin(2\pi b t_2)}{2\pi t_2} f_B(b) db$$

$$= \frac{4\sigma^2}{2\pi t_1 2\pi t_2}\int_{b_1}^{b_2}[\cos 2\pi b(t_1-t_2) - \cos 2\pi b(t_1+t_2)]\frac{1}{b_2-b_1} db$$

$$= \frac{\sigma^2}{\pi^2 t_1 t_2}[\sin c[(b_2-b_1)(t_1-t_2)]\cos\pi(b_2+b_1)(t_1-t_2) - \sin c[(b_2-b_1)(t_1+t_2)]\cos\pi(b_2+b_1)(t_1+t_2)] = R_{HH}(t_1,t_2) \tag{2.9}$$

Finally, from Eq. 2.6 obtain

$$S_{HH}^d(f,\nu) = \int_{-\infty}^{\infty} a^2 f_A(a) da \int_{-\infty}^{\infty} rect\left(\frac{f}{2b}\right) rect\left(-\frac{\nu}{2b}\right) f_B(b) db = E\{A^2\}\int_{b_1}^{b_2}\frac{1}{b_2-b_1} rect\left(\frac{f}{2b}\right) rect\left(-\frac{\nu}{2b}\right) db \tag{2.10}$$

The integral in Eq. 2.10 in spite of its apparent simplicity is not one that can be easily evaluated in closed form. However a useful approximation is still possible for $f = \nu$ (the case of interest) and using the fact that the rectangular function is even symmetric. Thus, if the integral is approximated by an N-point sum, then

$$S_{HH}^d(f,f) \simeq \frac{4\sigma^2}{b_2-b_1}\sum_{n=1}^{N} rect^2\left(\frac{f}{2b(n)}\right)\Delta_b \quad b(1)=b_1,\ b(N)=b_2,\ \Delta_b = (b_2-b_1)/N \tag{2.11}$$

It is not difficult to visualize the approximation of Eq. 2.11 since a summation of increasingly wide rectangular functions produces a “wedding cake” like shape for the frequency domain description of the channel. The base of the frequency dependent function has a width corresponding to the maximum possible value for the random variable B. These results even though specific to the assumed statistics of the channel gain and cutoff can be used to obtain the statistical description of the output process for an assumed model of the input process.

3. Example of a Practical Baseband Model

Consider now a more realistic low pass random channel model described by the impulse response

` $$H(t) = Ae^{-Bt}u(t) \tag{3.1}$$

having (as in Section 2 above) random gain and cutoff A and B, respectively. For such a channel,

$$E\{H(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ae^{-bt}u(t)f_{AB}(a,b)dadb \tag{3.2}$$

and

$$E\{H(t_1)H(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ae^{-bt_1}u(t_1)ae^{-bt_2}u(t_2)f_{AB}(a,b)dadb \tag{3.3}$$

Evaluation of these two expectations will be subsequently carried out using the assumed pdf of Eq. 2.7. Channel characterization in the frequency domain is most conveniently obtained from

$$S_{HH}^{d}(f,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ae^{-bt_1}u(t_1)ae^{-bt_2}u(t_2)f_{AB}(a,b)dadbe^{-j2\pi(ft_1-\nu t_2)}dt_1dt_2 \tag{3.4}$$

Integrating Eq. 3.4 over t_1 and t_2 first and using the Fourier Transform pair $e^{-bt}u(t)\longleftrightarrow\frac{1}{b+j2\pi f}$, $b>0$, yields

$$S_{HH}^{d}(f,\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a\frac{1}{b+j2\pi f}a\frac{1}{b-j2\pi\nu}f_{AB}(a,b)dadb \tag{3.5}$$

Using now Eq. 2.7 in order to obtain more specific results, obtain therefore from Eq. 3.2,

$$\begin{aligned} E\{H(t)\} &= \int_{-\infty}^{\infty} af_A(a)da\int_{-\infty}^{\infty} e^{-bt}u(t)f_B(b)db = E\{A\}\int_{b_1}^{b_2}\frac{1}{b_2-b_1}e^{-bt}u(t)db \\ &= E\{A\}u(t)\left[-\frac{e^{-bt}}{(b_2-b_1)t}\right]_{b_1}^{b_2} = \sqrt{\frac{\pi}{2}}\sigma\frac{e^{-b_1t}-e^{-b_2t}}{(b_2-b_1)t}u(t) = m_H(t) \end{aligned} \tag{3.6}$$

Similarly, from Eq. 3.3

$$E\{H(t_1)H(t_2)\} = \int_{-\infty}^{\infty} a^2 f_A(a)da\int_{-\infty}^{\infty} e^{-bt_1}u(t_1)e^{-bt_2}u(t_2)f_B(b)db$$

$$= \sigma^2\int_{b_1}^{b_2} e^{-b(t_1+t_2)}\frac{1}{b_2-b_1}dbu(t_1)u(t_2) = \frac{\sigma^2}{(b_2-b_1)}\left[\frac{e^{-b(t_1+t_2)}}{-(t_1+t_2)}\right]_{b_1}^{b_2}u(t_1)u(t_2) = \frac{\sigma^2}{b_2-b_1}\frac{e^{-b_1(t_1+t_2)}-e^{-b_2(t_1+t_2)}}{t_1+t_2}u(t_1)u(t_2) \tag{3.7}$$

Finally, from Eq. 3.5 obtain

$$\begin{aligned} S_{HH}^{d}(f,\nu) &= \int_{-\infty}^{\infty} a^2 f_A(a)da\int_{-\infty}^{\infty}\frac{1}{b+j2\pi f}\frac{1}{b-j2\pi\nu}f_B(b)db \\ &= E\{A^2\}\int_{b_1}^{b_2}\frac{1}{b_2-b_1}\frac{1}{b+j2\pi f}\frac{1}{b-j2\pi\nu}db = \frac{E\{A^2\}}{b_2-b_1}\int_{b_1}^{b_2}\frac{f}{f+\nu}\frac{1}{b(b+j2\pi f)}+\frac{\nu}{f+\nu}\frac{1}{b(b-j2\pi\nu)}db \end{aligned} \tag{3.8}$$

The integral of Eq. 3.8 can be evaluated using from [3] the closed form $\int\frac{dx}{x(\alpha+\beta x)} = -\frac{1}{\alpha}\ln\frac{\alpha+\beta x}{x}$. Therefore,

$$S_{HH}^{d}(f,\nu) = \frac{E\{A^2\}}{b_2-b_1}\left[-\frac{f}{f+\nu}\frac{1}{j2\pi f}\ln\frac{j2\pi f+b}{b}+\frac{\nu}{f+\nu}\frac{1}{j2\pi\nu}\ln\frac{-j2\pi\nu+b}{b}\right]_{b_1}^{b_2}$$

$$= \frac{E\{A^2\}}{j2\pi(f+\nu)(b_2-b_1)}\left[\ln\frac{j2\pi f+b_1}{b_1}\frac{b_2}{j2\pi f+b_2}+\ln\frac{-j2\pi\nu+b_2}{b_2}\frac{b_1}{-j2\pi\nu+b_1}\right]$$

$$= \frac{E\{A^2\}}{j2\pi(f+\nu)(b_2-b_1)}\left[\ln\frac{2\pi f-jb_1}{2\pi f-jb_2}\frac{2\pi\nu+jb_2}{2\pi\nu+jb_1}\right] \tag{3.9}$$

Define for convenience (using both rectangular as well as polar notation)

$$g_i(\vartheta)=2\pi\vartheta+jb_i=G_i(\vartheta)e^{j\alpha_i(\vartheta)}=\sqrt{(2\pi\vartheta)^2+b_i^2}\,e^{j\tan^{-1}(b_i/2\pi\vartheta)} \qquad i=1,2 \tag{3.10}$$

so that the logarithmic expression inside the brackets in Eq. 3.9 can be expressed as

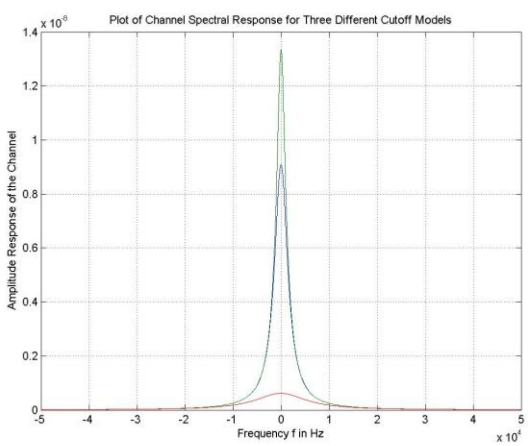
$$\ln\frac{g_1^*(f)g_2(\nu)}{g_2^*(f)g_1(\nu)}=\ln\frac{G_1(f)e^{-j\alpha_1(f)}G_2(\nu)e^{j\alpha_2(\nu)}}{G_2(f)e^{-j\alpha_2(f)}G_1(\nu)e^{j\alpha_1(\nu)}}=\ln\frac{G_1(f)G_2(\nu)}{G_2(f)G_1(\nu)}+j[-\alpha_1(f)+\alpha_2(\nu)+\alpha_2(f)-\alpha_1(\nu)] \tag{3.11}$$

Therefore,

$$S_{HH}^d(f,\nu)=\frac{E\{A^2\}}{j2\pi(f+\nu)(b_2-b_1)}\{\ln\frac{G_1(f)G_2(\nu)}{G_2(f)G_1(\nu)}+j[-\alpha_1(f)+\alpha_2(\nu)+\alpha_2(f)-\alpha_1(\nu)]\} \tag{3.12}$$

The specific case of interest, namely $S_{HH}^d(f,\nu)$ with $f=\nu$ follows from Eq. 3.12 and simplifies to

$$S_{HH}^d(f,f)=\frac{E\{A^2\}}{2\pi f(b_2-b_1)}[-\alpha_1(f)+\alpha_2(f)]=\frac{E\{A^2\}}{2\pi f(b_2-b_1)}\left[-\tan^{-1}\frac{b_1}{2\pi f}+\tan^{-1}\frac{b_2}{2\pi f}\right] \tag{3.13}$$



It is a simple matter to plot the result of Eq. 3.13 for representative values of b_1 and b_2. This has been done for three sets of values, namely, $(b_1=10000, b_2=11000)$, $(b_1=5000, b_2=15000)$, and $(b_1=40000, b_2=40500)$. The three plots depicting $S_{HH}^d(f,f)$ are shown on the left.

4. Output Statistics Pertaining to the Channel Model Presented in Section 2

Consider a channel input consisting of random binary data.

References

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