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The Logarithmic Spiral: Mathematical Properties and Turbulence

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Abstract

We consider mathematical properties of the logarithmic spiral and its use in modeling turbulence. We analyze mathematically the set of point crossings resulting from a linear intersection through the center of a logarithmic spiral. We derive analytically the fractal dimension as a function of scale for this set of crossings. We also derive analytically the power spectrum of the thresholded function corresponding to these point crossings. These results have implications for turbulence modeling which we discuss in the context of experimental observations of logarithmic spiral structures of scalar fields in turbulence.

1. Introduction

Spirals are widely observed in a broad range of phenomena including galaxies, biological organisms, atmospheric flows, as well as turbulence (Catrakis 2000, Aldridge 1998, Moffatt 1993, Gilbert 1988, Burton 1973, Castle 1934). In turbulence, logarithmic spirals have been observed for scalar fields (Everson & Sreenivasan 1992) and algebraic spirals have been observed for vortex fields (Moffatt 1993, Angilella & Vassilicos 1999). Whereas algebraic spirals have been studied extensively before (Gilbert 1988, Moffatt 1993, Vassilicos & Brasseur 1996), we will focus on logarithmic spirals whose fractal and spectral aspects have not been studied in detail before to our knowledge.

In this work, we consider analytically the set of point crossings in a linear intersection of the logarithmic spiral through its center. In section 2, we analytically explore the crossing scales, i.e. the spacing scales between successive point crossings, in order to obtain the probability density function of crossing scales. This enables us to derive the fractal dimension as a function of scale for the point crossings of the logarithmic spiral. Also in section 2, we derive the power spectrum of the thresholded function corresponding to the point crossings. In addition, we discuss the implications of our results for turbulence modeling. We summarize in section 3 our conclusions.

2. Mathematical Analysis of the Logarithmic Spiral

The logarithmic spiral can be expressed in polar coordinates, or equivalently in other forms such as Cartesian coordinates or parametric coordinates, as follows:

$$\begin{aligned} r &= be^{-a\theta} \\ x^2 + y^2 &= b^2 e^{-2a\tan^{-1}(y/x)} \\ x &= be^{-a\theta}\cos\theta \\ y &= be^{-a\theta}\sin\theta \end{aligned}$$

where we will choose the angle to be increasing from zero in order to ensure that the spiral converges to the origin and we choose the parameter b to be positive. An example of the logarithmic spiral is shown in figure 1.

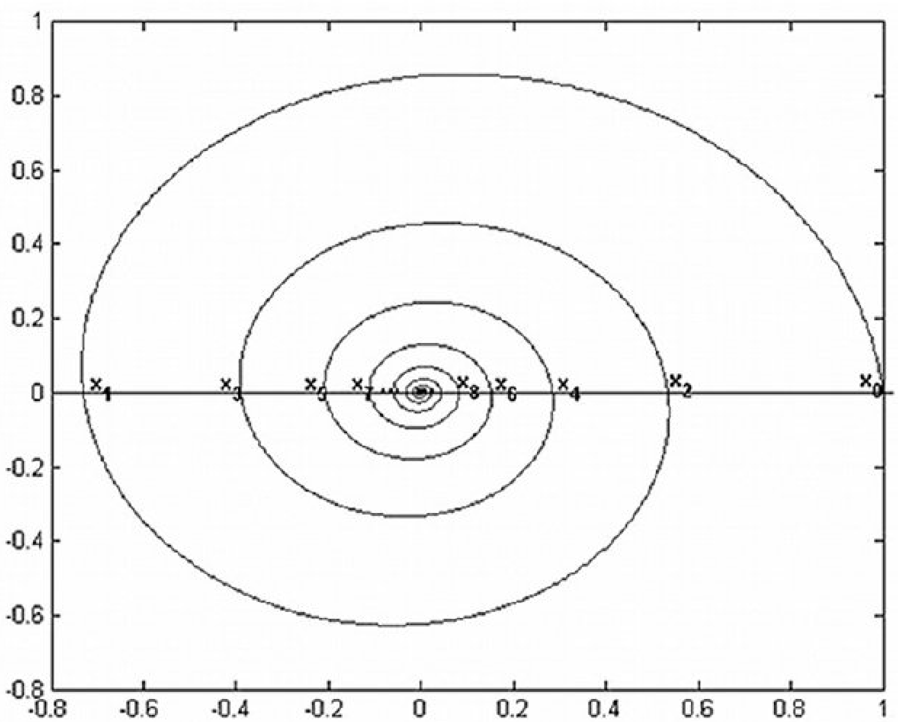

Figure 1. Example of a logarithmic spiral and the set of crossing points formed by the intersections of the spiral through its center along the horizontal axis. Successive crossings are labeled as x_k with $k = 0, \dots, N$, where in the limit $N \to \infty$.

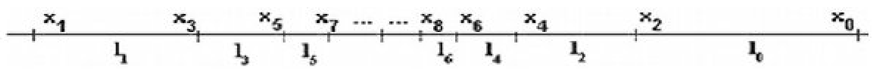

Figure 2. Schematic of the crossing scales l_k, i.e. the spacings between pairs of successive crossing points x_k and x_{k+2}, along the intersection through the center of the logarithmic spiral.

If we intersect the spiral through its center along the horizontal axis, the intersections of the spiral will occur at the following angles with the following horizontal locations of these crossing points and corresponding spacings between successive crossing points, as follows:

$$\theta_k = k\pi$$

$$x_k = (-1)^k b e^{-a\theta_k} = (-1)^k b e^{-ak\pi}$$

$$|x_k| = b e^{-a\theta_k}$$

$$l_k = |x_k - x_{k+2}| = (-1)^k (x_k - x_{k+2}) = b(e^{-ak\pi} - e^{-a(k+2)\pi})$$

where the reason the pairs of indices are k and k+2 is because every pair of successive crossing points occurs on either the right or the left side of the spiral as in figure 1. We show in figure 2 a schematic of the crossing scales and the corresponding pairs of successive crossing points along the intersection through the center of the spiral.

We can now use the general relation derived by the author (Catrakis 2000) which relates directly the probability density function (pdf) of crossing scales to the fractal dimension as a function of the coverage scale, i.e. the partition scale corresponding to successively partitioning the entire interval containing all the point crossings. For the logarithmic spiral, the pdf of the crossing scales can be written as a sum of delta functions because all of the crossing scales in have distinct values, with the understanding that the analysis of the crossing scales for the entire logarithmic spiral corresponds to the limit N increasing to infinity. We plot the dimension in figure 3 where we see that there is an overall increase in the dimension with increasing scale, i.e. there is no constant dimension, and moreover there is non-monotonicity associated with the consecutive transitions between successive parts of the spiral at its intersection with the horizontal axis as can be seen in figure 1. The logarithmic spiral thus offers one possible model for such observed behavior and it is a particularly plausible model because there are direct experimental observations of logarithmic spirals found in scalar fields in turbulence (Everson & Sreenivasan 1992). This is a useful result in the context of turbulence because there are various observations such as experimental studies by several authors cited in the work of Catrakis (2000) which show an overall increase of the dimension with increasing scale.

Our mathematical derivation therefore for the dimension is as follows:

$$p(l) = \frac{1}{N}\sum_{k=0}^{N-1}\delta(l - l_k)$$

$$D(\lambda) = 1 - \frac{\lambda\int_\lambda^\infty p(l)dl}{\int_0^\lambda\int_{\lambda'}^\infty p(l)dld\lambda'}$$

$$D(\lambda) = 1 - \frac{\lambda A(\lambda)}{B(\lambda)}$$

$$A(\lambda) = \int_\lambda^\infty p(l)dl = \begin{cases} 0 \quad , \quad \lambda > l_0 \\ \frac{k}{N} \quad , \quad l_{k-1} > \lambda > l_k \quad , \quad k = 1, 2, \ldots, N-1 \\ 1 \quad , \quad \lambda < l_{N-1} \end{cases}$$

$$B(\lambda) = \int_0^\lambda\int_{\lambda'}^\infty p(l)dld\lambda' = \begin{cases} \frac{1}{N}\sum_{k'=1}^{N-1}k'(l_{k'-1} - l_{k'}) + l_{N-1} \quad , \quad \lambda > l_0 \\ \frac{k}{N}(\lambda - l_k) + \frac{1}{N}\sum_{k'=k+1}^{N-1}k'(l_{k'-1} - l_{k'}) + l_{N-1} \quad , \quad l_{k-1} > \lambda > l_k \quad , \quad k = 1, 2, \ldots, N-2 \\ \frac{N-1}{N}(\lambda - l_{N-1}) + l_{N-1} \quad , \quad l_{N-2} > \lambda > l_{N-1} \\ \lambda \quad , \quad \lambda < l_{N-1} \end{cases}$$

$$D(\lambda) = \int_0^\lambda\int_{\lambda'}^\infty p(l)dld\lambda' = \begin{cases} 1 \quad , \quad \lambda > l_0 \\ 1 - \dfrac{\lambda\dfrac{k}{N}}{\dfrac{k}{N}(\lambda - l_k) + \dfrac{1}{N}\displaystyle\sum_{k'=k+1}^{N-1}k'(l_{k'-1} - l_{k'}) + l_{N-1}} \quad , \quad l_{k-1} > \lambda > l_k \quad , \quad k = 1, 2, \ldots, N-2 \\ 1 - \dfrac{\lambda\left(\dfrac{N-1}{N}\right)}{\dfrac{(N-1)}{N}(\lambda - l_{N-1}) + l_{N-1}} \quad , \quad l_{N-2} > \lambda > l_{N-1} \\ 0 \quad , \quad \lambda < l_{N-1} \end{cases}$$

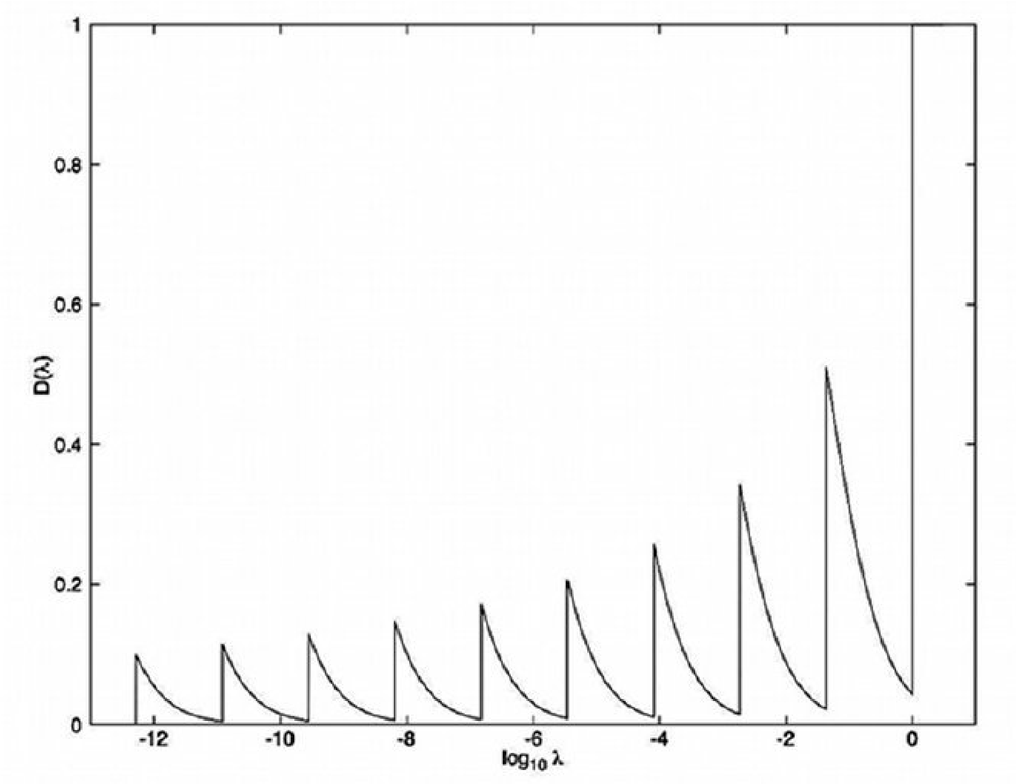

Figure 3. The dimension $D(\lambda)$ of the logarithmic spiral plotted as a function of logarithmic scale.

Also, we derive analytically the power spectrum for a thresholded function corresponding to the logarithmic spiral. We analytically derive the power spectral density of our thresholded function by taking the square of the modulus of the Fourier transform of the thresholded function, i.e. by taking the product of the Fourier transform and its complex conjugate.

We achieve this by expressing the thresholded function in terms of the finite-width step function because this helps us to take analytically the Fourier transform of the thresholded function in terms of a linear superposition of the individual Fourier transforms of the finite-width step functions.

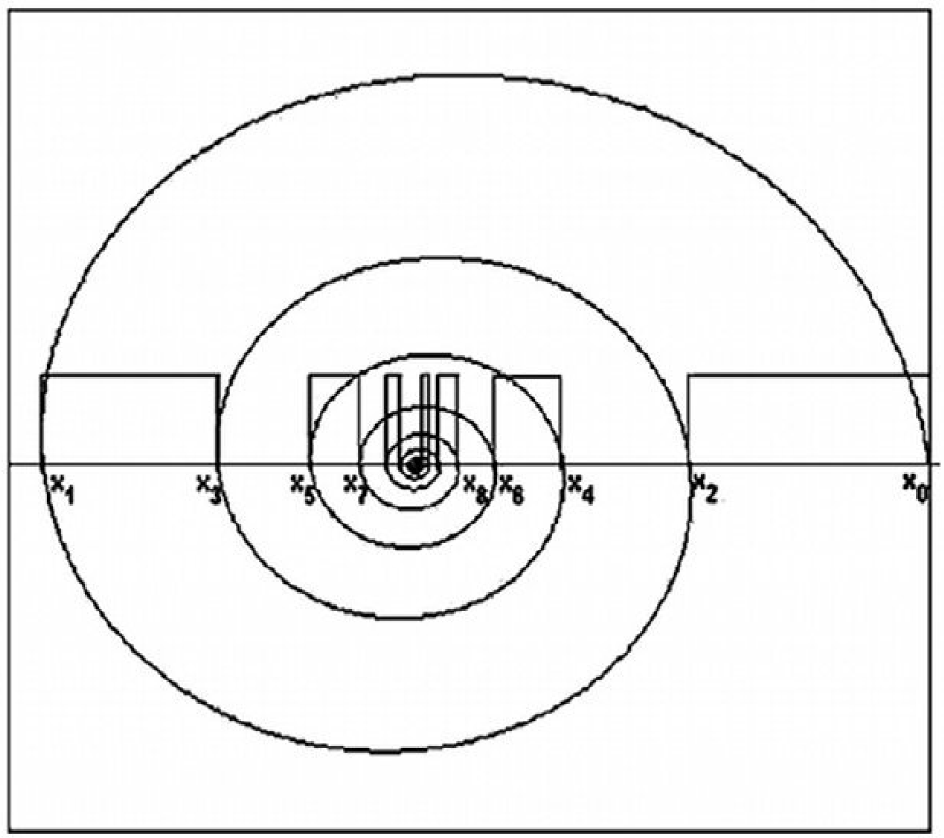

Figure 4. Thresholded function created in terms of step functions between successive pairs of point crossings of the logarithmic spiral through its center along the horizontal axis.

As shown in figure 4, we have constructed a thresholded function as a summation series of step functions such that the thresholded function has steps at successive pairs of point crossings of the logarithmic spiral through its center along the horizontal axis. Thus, we write analytically the thresholded function in terms of a first pair of Heaviside functions corresponding to pairs of crossings at positive locations and a second pair of Heaviside functions corresponds to pairs of crossings at negative locations, on the horizontal axis.

Our mathematical derivation for the power spectrum is therefore as follows:

$$f(x) = \sum_{k=0}^{\infty} \left[[H(x - x_{4k+2}) - H(x - x_{4k})] + [H(x - x_{4k+1}) - H(x - x_{4k+3})] \right]$$

$$f(x) = \sum_{k=0}^{\infty} \left[W_{x_{4k} - x_{4k+2}}(x - x_{4k+2}) + W_{x_{4k+3} - x_{4k+1}}(x - x_{4k+1}) \right]$$

$$W_a(x-b) = H(x-b) - H(x-b-a) = \begin{cases} 0 \ , & x < b \\ 1 \ , & b \le x < b+a \\ 0 \ , & x \ge b+a \end{cases}$$

$$\hat{W}_{a,b}(\omega) = \frac{1}{\sqrt{2\pi}} \int_b^{b+a} e^{-i\omega x} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-i\omega b} - e^{-i\omega(b+a)}}{i\omega} = \sqrt{\frac{2}{\pi}} e^{-i\omega\left(b+\frac{a}{2}\right)} \frac{\sin(\omega a/2)}{\omega}$$

$$\begin{aligned}
\left|\hat{f}(\omega)\right|^2 &= \hat{f}(\omega)\hat{f}^*(\omega) \\
&= \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \left[e^{-i\frac{x_{4k}+x_{4k+2}}{2}\omega} \frac{\sin\left(\frac{l_{4k}}{2}\omega\right)}{\omega} + e^{-i\frac{x_{4k+3}+x_{4k+1}}{2}\omega} \frac{\sin\left(\frac{l_{4k+1}}{2}\omega\right)}{\omega} \right] \\
&\quad \times \sqrt{\frac{2}{\pi}} \sum_{k'=0}^{\infty} \left[e^{i\frac{x_{4k'}+x_{4k'+2}}{2}\omega} \frac{\sin\left(\frac{l_{4k'}}{2}\omega\right)}{\omega} + e^{i\frac{x_{4k'+3}+x_{4k'+1}}{2}\omega} \frac{\sin\left(\frac{l_{4k'+1}}{2}\omega\right)}{\omega} \right] \\
&= \frac{2}{\pi} \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \left\{ \left[e^{-i\frac{x_{4k}+x_{4k+2}}{2}\omega} \left(\frac{\sin\left(\frac{l_{4k}}{2}\omega\right)}{\omega} \right) + e^{-i\frac{x_{4k+3}+x_{4k+1}}{2}\omega} \left(\frac{\sin\left(\frac{l_{4k+1}}{2}\omega\right)}{\omega} \right) \right] \right. \\
&\qquad \left. \times \left[e^{i\frac{x_{4k'}+x_{4k'+2}}{2}\omega} \left(\frac{\sin\left(\frac{l_{4k'}}{2}\omega\right)}{\omega} \right) + e^{i\frac{x_{4k'+3}+x_{4k'+1}}{2}\omega} \left(\frac{\sin\left(\frac{l_{4k'+1}}{2}\omega\right)}{\omega} \right) \right] \right\}
\end{aligned}$$

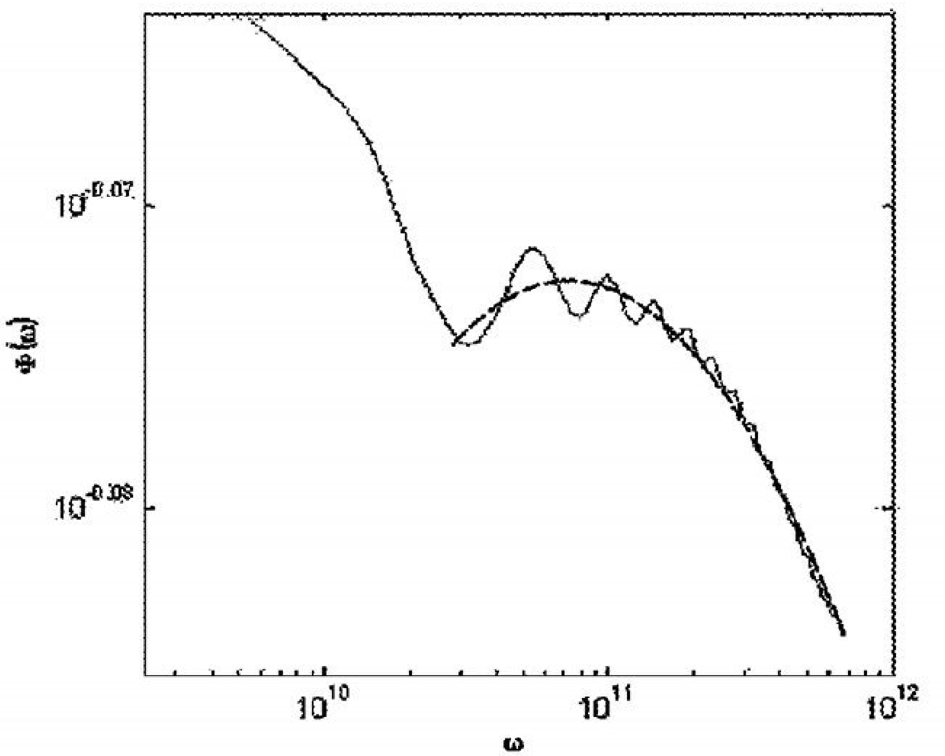

Figure 5. Log-log plot of the analytical power spectrum (solid curve) and comparison to a parabola (dashed curve) which corresponds to the lognormal power spectrum model.

We show in figure 5 the power spectrum for the thresholded function corresponding to the point crossings of the logarithmic spiral. A parabolic fit in log-log coordinates is also shown as a dashed curve and gives a good fit at high frequencies. The parabola is of interest because a parabolic behavior would mean that the power spectrum is approximately lognormal.

Our analysis of the logarithmic spiral agrees with observations of scalar fields in turbulence. There are empirical observations and measurements of scalar fluctuations in turbulence (e.g. Everson & Sreenivasan 1992, Miller & Dimotakis 1996) which indicate a lognormal power spectrum in linear-linear coordinates, i.e. a parabolic power spectrum in log-log coordinates. Thus, our present analysis offers a theoretical explanation of such experimental findings and suggests that logarithmic spirals have an important role as physical structures for mathematical modeling of scalar fields in turbulence.

3. Conclusions

We have presented an analysis of mathematical properties of the logarithmic spiral. An important motivation in turbulence for studying logarithmic spirals is that there are experimental observations of logarithmic spiral structures of scalar fields in turbulence. By considering mathematically the set of point crossings resulting from a linear intersection through the center of a logarithmic spiral, we derived analytically the fractal dimension as a function of scale for this set of crossings. We also derived analytically the power spectrum of the thresholded function corresponding to these point crossings. In the context of turbulence, these results are useful because they provide a mathematical model that can be utilized in studies of the behavior of scalar fields in turbulent flows.

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