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Mathematical Aspects of Level Crossings of Multiscale Functions with Implications in Turbulence

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Abstract

We consider mathematical aspects of level crossings of fluctuating functions with emphasis on spectral and fractal properties. A general approach for relating spectra and fractals is established by combining relations for the power spectrum and for the fractal dimension, respectively, to the probability density function of level crossing scales. These relations indicate that a given power spectrum can be non-uniquely related to different probability density functions of level crossing scales. As an illustrative example of this non-uniqueness, we explore two functions which have the same power-law spectral slope and yet have different level crossing statistics. Implications of this general mathematical approach for turbulent flows are discussed in the context of turbulence theories and observations.

1. Introduction

Level crossings of fluctuating multiscale functions are important in various phenomena including turbulence (Catrakis 2000; Edwards & Hurst 2001; Fuchikami & Ishioka 2004; Kailasnath & Sreenivasan 1993; Mandelbrot 1975; Shimokawa & Ozawa 2005; Sreenivasan & Bershadskii 2006; Sreenivasan et al. 1983; Vassilicos & Brasseur 1996; Yee et al. 1995). By level crossings, we denote the locations where the function attains a given threshold value. Examples of practically important cases are level crossings of temperature for global climate change studies, level crossings of wind speed for wind energy applications, or level crossings of turbulence-aberrated optical intensity for laser propagation applications. A statistical quantity of basic interest is the probability density function (pdf) of level crossings, i.e. the pdf of the scales between successive level crossings. In various turbulent flows and for a variety of flow variables such as velocity or concentration, it is of both fundamental and practical interest to investigate the statistics of the level crossings, i.e. whether the level crossings pdfs are power laws, exponentials, or lognormals for example (Catrakis 2000).

In this paper, we are interested in exploring the mathematical relations between the pdf of level crossings, the power spectrum, and the fractal dimension. All three quantities are often studied in turbulence (e.g. Catrakis 2000; Edwards & Hurst 2001; Sreenivasan & Bershadskii 2006; Vassilicos & Brasseur 1996) and yet there has not been a focus previously on exploring the relations between all three quantities. In section 2, we will explore a general mathematical approach for relating those three quantities and we will see that these relations suggest that a given power spectrum can be non-uniquely related to different pdfs of level crossing scales. In section 3, we explore an illustrative example of this non-uniqueness using two functions which have the same power-law spectral slope but have different level crossing statistics. We discuss the implications of these findings for turbulenc and in section 4 we summarize our conclusions.

2. Theoretical Aspects of Level Crossings for Relating Spectra and Fractals

Consider a fluctuating function *f(t)* and the level crossings of this function for a given threshold as shown in the example of figure 1. We note that the function can be a function of time or space, or even a function of several space-time variables, and for simplicity of notation we consider it here as a real function of time. Let us consider the probability density function (pdf) of level crossing scales, i.e. the time intervals between successive level crossings, and let us denote this pdf of level crossings as $p(\tau)$ which therefore is normalized as:

$$\int_0^\infty p(\tau)\, d\tau = 1 \quad (1)$$

There is an exact mathematical relation that has been derived by the author (Catrakis 2000) between the level crossings pdf $p(\tau)$ and the fractal dimension of the level crossings where in this analysis the fractal dimension is the coverage dimension based on partitioning a time interval into successively smaller scales. These level crossings are the set of points which are the locations of the crossings of a given threshold as denoted for example by square symbols in figure 1. The result of the analysis by Catrakis (2000) is that the fractal dimension $D(\tau)$ as a function of scale τ is expressed in terms of the level crossings pdf $p(\tau)$ as:

$$D(\tau) = 1 - \frac{\tau \int p(\tau')\, d\tau'}{\int \int p(\tau'')\, d\tau'' d\tau'} \quad (2)$$

This relation is invertible and so we can express the level crossings pdf $p(\tau)$ in terms of the fractal dimension $D(\tau)$ as:

$$p(\tau) = \frac{\tau_m}{\tau^2}\left\{D(\tau)\left[1 - D(\tau)\right] + \tau \frac{dD(\tau)}{d\tau}\right\} \times \exp\left\{-\int \left[1 - D(\tau')\right]\frac{d\tau'}{\tau'}\right\} \quad (3)$$

where the mean level crossing scale τ_m is written as:

$$\left\{\tau \exp\left\{-\int \left[1 - D(\tau')\right]\frac{d\tau'}{\tau'}\right\}\right\} \quad (4)$$

The above general relations can be applied therefore to various cases of level crossing statistics such as power law, exponential, lognormal, or any level crossing pdfs (Catrakis 2000).

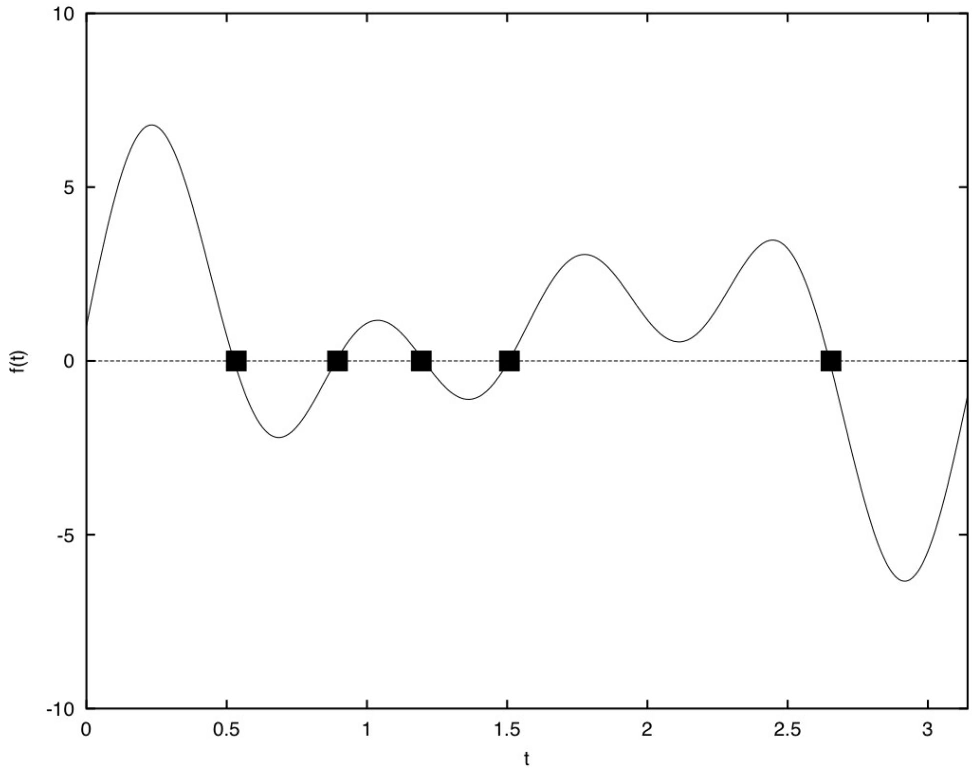

Figure 1: Example of a fluctuating function *f(t)* and its level crossings at threshold $f_* = 0$, where the level crossings are denoted as the square symbols.

Let us now also consider the thresholded function *h(t)* which we define so that it has values +1 or -1 according to whether the function *f(t)* is greater than or equal to a given threshold or less than a given threshold respectively:

$$h(t) = +1,\ f(t) \geq f_* \\ h(t) = -1,\ f(t) < f_* \quad (5)$$

An example of the thresholded function *h(t)* is shown in figure 2. In the literature, this thresholded function is sometimes referred to as the dichotomous function, e.g. in the study by Fuchikami & Ishioka (2004). Other studies sometimes use +1 and 0 as the two values of the thresholded function and denote it as the telegraph function, e.g. in the study by Sreenivasan & Bershadskii (2006).

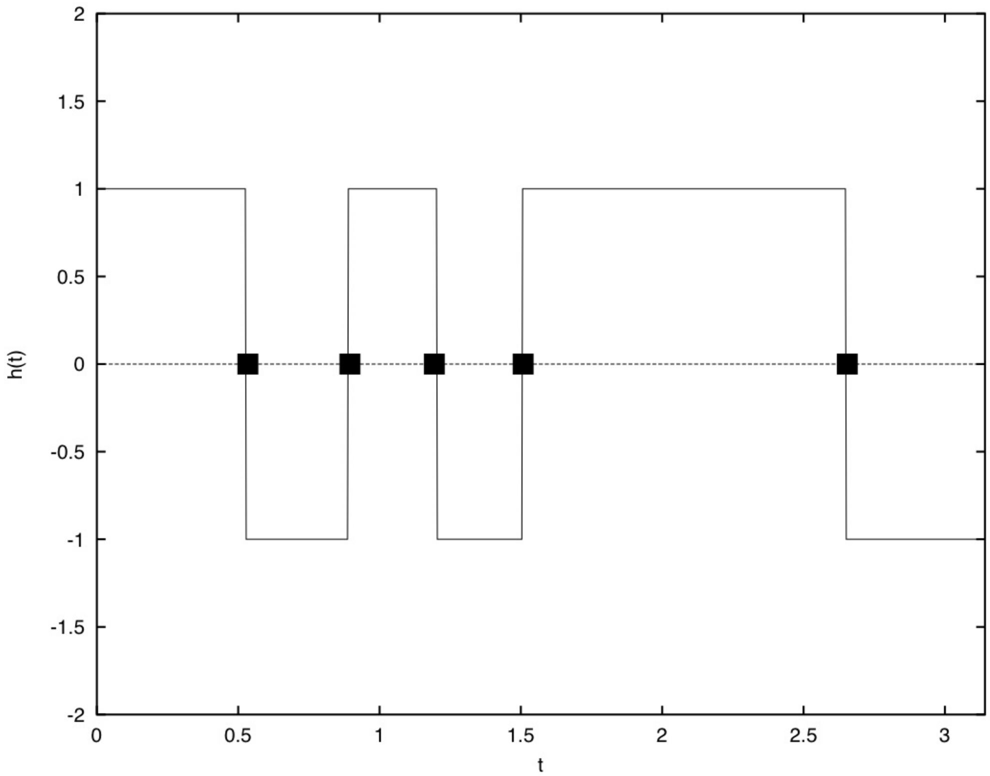

Figure 2: Example of the thresholded function *h(t)*, corresponding to the original function *f(t)* from figure 1, and its level crossings at zero threshold denoted as the square symbols.

Let us now consider the autocorrelation function *g(τ)* of the thresholded function *h(t)* which is therefore:

$$g(\tau) = h(t)\,h(t+\tau) \qquad (6)$$

There is also an exact relation between the level crossings pdf *p(τ)* and the autocorrelation function *g(τ)* of the thresholded function *h(t)*, as derived by Fuchikami & Ishioka (2004) and this is expressed as follows:

$$g(\tau) = \frac{1}{\tau} - \frac{2}{\tau_m \tau^2}\frac{1-\int e^{-\tau\tau'} p(\tau')\,d\tau'}{1+\int e^{-\tau\tau'} p(\tau')\,d\tau'} = \frac{1}{\tau} - \frac{2}{\tau_m \tau^2}\frac{1-L\{p(\tau')\}}{1+L\{p(\tau')\}} \qquad (7)$$

where the integrals are Laplace transforms denoted as $L\{\}$. One can invert this relation in order to express the level crossings pdf *p(τ)* in terms of the autocorrelation function *g(τ)* as follows:

$$p(\tau) = L^{-1}\left\{ \frac{1-\frac{\tau m}{2}\tau'[1-\tau' g(\tau')]}{1+\frac{\tau m}{2}\tau'[1-\tau' g(\tau')]} \right\} \quad (8)$$

where L^{-1} is the inverse Laplace transform.

We can now accomplish the main step in our present paper which is to relate the power spectrum *S(ω)* to the level crossings pdf *p(τ)* and, therefore using equation 2, we can also now relate the power spectrum *S(ω)* to the fractal dimension *D(τ)* as a function of scale *τ*. We achieve this by recalling that the power spectrum *S(ω)* is the square of the magnitude of the Fourier transform of the autocorrelation function *h(τ)* , i.e $S(\omega)=|F\{g(\tau)\}|^2$. Therefore the power spectrum *S(ω)* is related to the level crossings pdf *p(τ)* using the following equation:

$$S(\omega) = \left(F\left\{ \frac{1}{\tau} - \frac{2}{\tau_m \tau^2} \frac{1-L\{p(\tau')\}}{1+L\{p(\tau')\}} \right\}^2 \right) \quad (9)$$

which involves Fourier and Laplace transforms. In addition, we see that the power spectrum *S(ω)* is related to the fractal dimension *D(τ)* as a function of scale *τ* using equations 3 and 9 so that we have this relation:

$$S(\omega) = \left(F\left\{ \frac{1}{\tau} - \frac{2}{\tau_m \tau^2} \frac{1-L\left\{ \frac{\tau_m}{\tau'^2}\left\{ D(\tau')[1-D(\tau')]+\tau'\frac{dD(\tau')}{d\tau'} \right\} e^{\left\{-\int [1-D(\tau'')]\frac{d\tau''}{\tau''}\right\}} \right\}}{1+L\left\{ \frac{\tau_m}{\tau'^2}\left\{ D(\tau')[1-D(\tau')]+\tau'\frac{dD(\tau')}{d\tau'} \right\} e^{\left\{-\int [1-D(\tau'')]\frac{d\tau''}{\tau''}\right\}} \right\}} \right\}^2 \right) \quad (10)$$

which also involves Fourier and Laplace transforms. Therefore, with equations 2, 3, 9, and 10, we have general mathematical relations between the level crossings pdf *p(τ)*, the fractal dimension *D(τ)*, and the power spectrum *S(ω)*.

It is important to realize in equations 9 and 10 that uniqueness is not guaranteed in terms of the inverting these equations, i.e. equations 9 and 10 are not invertible. This is because the magnitude operators in equations 9 and 10 remove all Fourier phase information. This raises the possibility for example that two functions with different level crossing statistics can have the same power spectrum. In the next section, we will explore this key issue of non-uniqueness with an example for the case of power-law statistics and implications for turbulence studies.

3. Application to Power-Law Spectrum with Non-Unique Level Crossing Pdfs and Implications for Turbulence

As an illustrative application of the above mathematical approach and especially of the possible non-uniqueness of level crossing pdfs, we explore two exact analytical functions which have the same power spectral slope and yet have different level crossing statistics. Our motivation in this section is the study of Vassilicos & Brasseur (1996) who considered several functions with the same power spectrum. However, Vassilicos & Brasseur (1996) did not analyze the level crossing statistics. So, the new contribution in our present study is to examine the level crossing statistics in relation to the power spectrum.

For the first function, let us consider a function which has increasingly higher frequencies at shorter times so that it has an accumulating singularity in the limit of $t \rightarrow 0^+$. Such a function is given by:

$$f(t) = sin\left(\frac{1}{\sqrt{t}}\right) \qquad (11)$$

for $t>0$. We show in figure 3 a plot of this function and we also show in figure 4 a plot of the thresholded function $h(t)$ for zero threshold. The level crossings in figure 4 show that there are successively smaller level crossing scales at shorter times. For $t>0$, the zeroes of the function $f(t)$ are given by:

$$t_k = \frac{1}{(k\pi)^2} \qquad (12)$$

where $k = 1, 2, 3, \ldots, \infty$. Therefore, the level crossing scales are as follows:

$$\tau_k = t_k - t_{k+1} = \frac{1}{(k\pi)^2} - \frac{1}{[(k+1)\pi]^2} = \frac{2k+1}{[k(k+1)\pi]^2} \qquad (13)$$

where $k = 1, 2, 3, \ldots, \infty$. Because in this case all the level crossing scales are distinct, we can write therefore the level crossing pdf $p(\tau)$ as a normalized sum of delta functions as follows:

$$p(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \delta\left(\tau - \frac{2k+1}{\left[k(k+1)\pi\right]^2}\right) \tag{14}$$

Thus, we could in principle substitute equation 14 which defines the level crossing pdf into equations 3 and 9 to obtain analytically the fractal dimension and the power spectrum. Rather than doing so here, we will focus on comparing computationally the plots of the level crossing pdf and power spectrum of this function as well as a different function below as we are mainly interested in illustrating the idea of non-uniqueness of the level crossing pdfs.

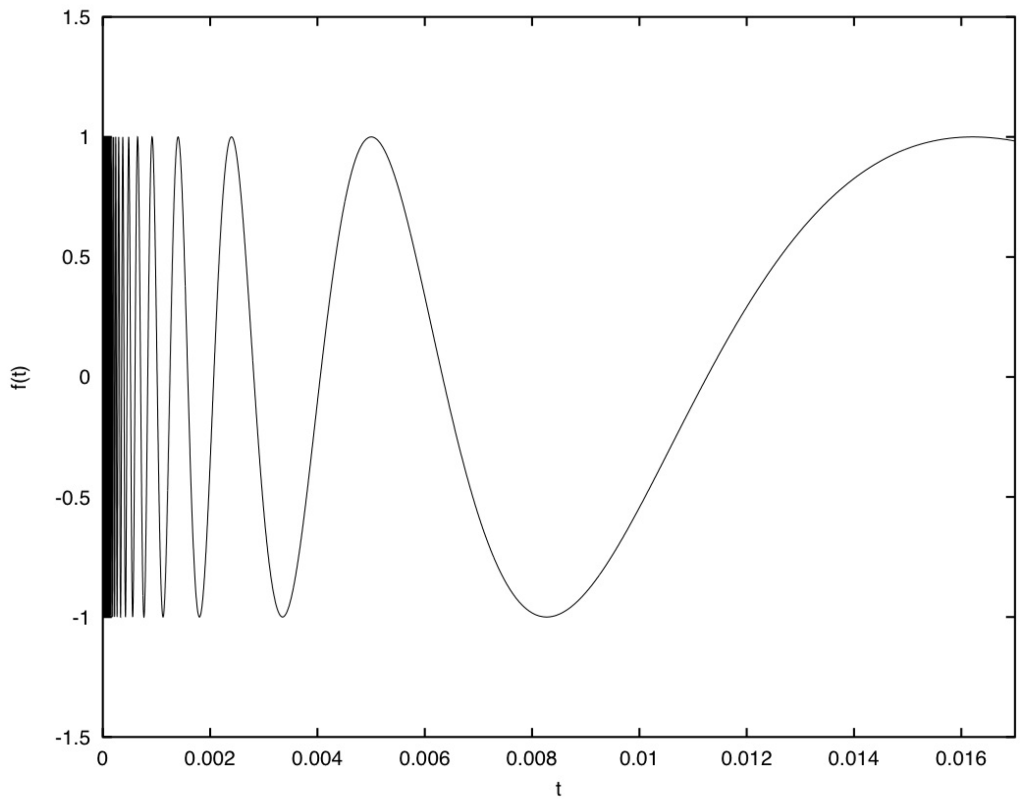

Figure 3: Plot of the accumulating singularity function of *f(t)* from equation 11 showing the presence of higher frequencies at shorter times.

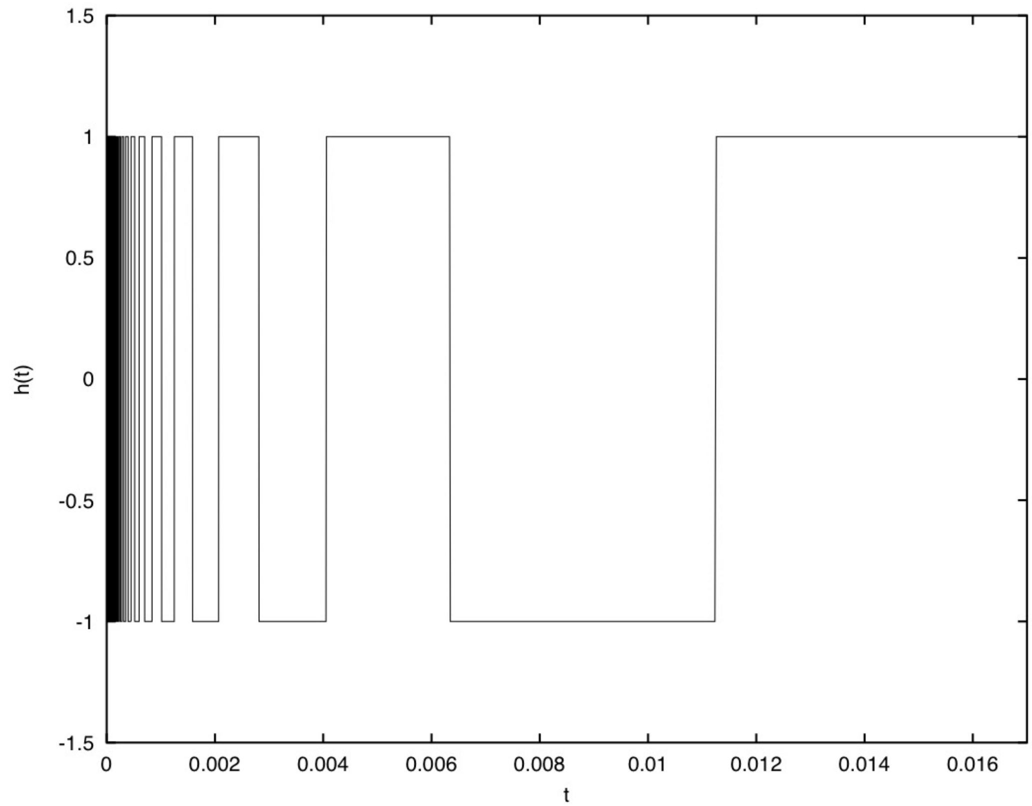

Figure 4: Plot of the thresholded function *h(t)* showing the level crossings of the accumulating singularity function of *f(t)* from equation 11 at zero threshold.

For the second function, let us consider the Weierstrass function which has various frequencies at many different times. The Weierstrass function can be written as:

$$f(t) = \sum_{k=0}^{\infty} a^{bk} sin(a^k t) \tag{15}$$

where in this study we will choose the parameters as a=1.2 and b=-5/6, for t>0. We show in figure 5 a plot of the Weierstrass function and we also show in figure 6 a plot of the thresholded function *h(t)* for zero threshold. The level crossings in figure 6 show that there are widely varying level crossing scales at various times. In contrast to the accumulating-singularity function in equation 11 and its analytically-derived level crossing pdf in equation 14, it does not appear straightforward to derive analytically the zeroes of the Weierstrass function in equation 15 and thus we will focus here on computational plots of the level crossing pdfs and of the corresponding power spectra.

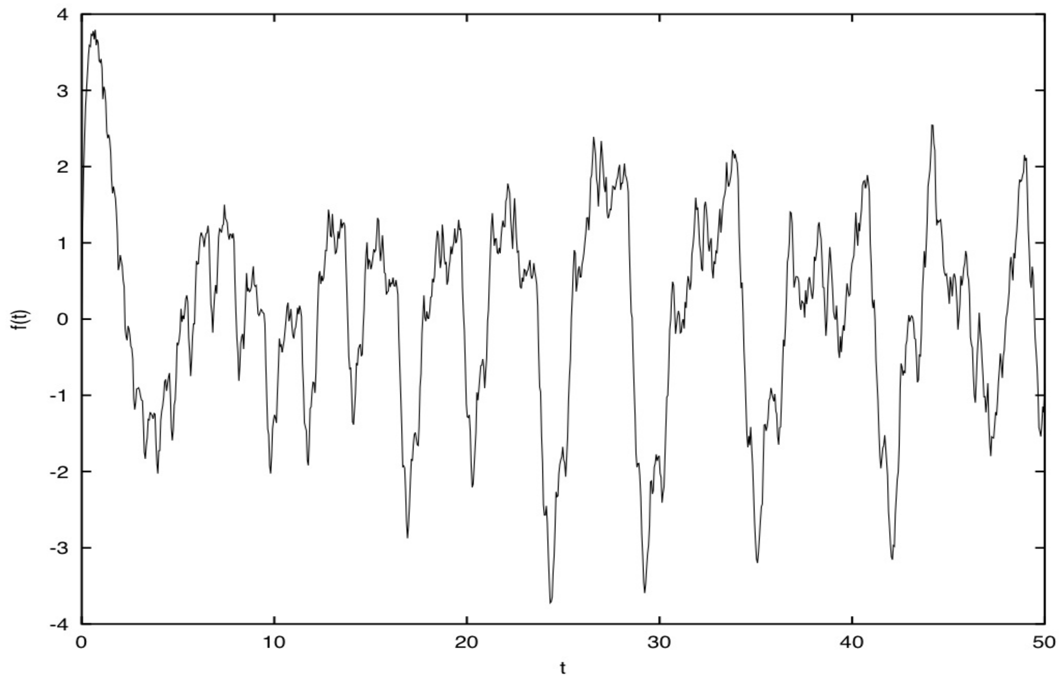

Figure 5: Plot of the Weierstrass function of *f(t)* from equation 15 showing the presence of multiple scales.

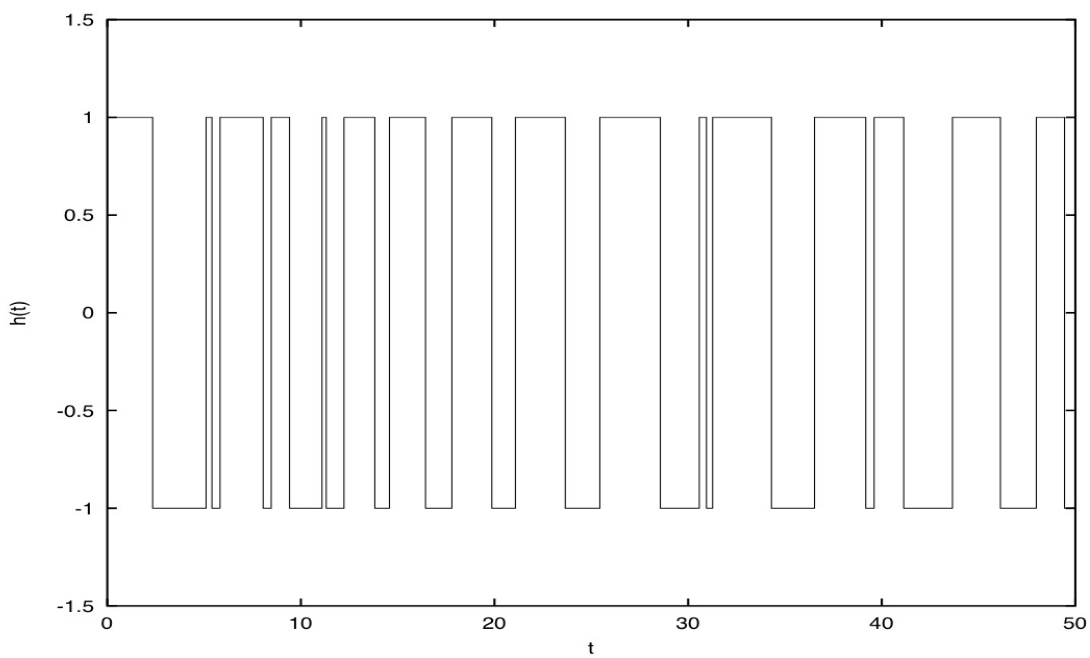

Figure 6: Plot of the thresholded function *h(t)* showing the level crossings of the Weierstrass function of *f(t)* from equation 15 at zero threshold.

Both of the above functions, i.e. the accumulating-singularity function in equation 11 and the Weierstrass function in equation 15, have power spectra with the same spectral-scaling slope of -5/3 which is an important spectral slope in the context of turbulent flows as there are many observations and computations of -5/3 spectral scaling, i.e. Kolmogorov scaling, at large Reynolds numbers in the inertial range of turbulent scales. Figure 7 shows the power spectrum of the accumulating-singularity function in log-log coordinates. The dotted line is the -5/3 spectral slope. Figure 8 shows the power spectrum of the Weierstrass function in log-log coordinates. The dotted line again is the -5/3 spectral slope. Both figures 7 and 8 show that both functions have the same spectral-scaling slope of -5/3.

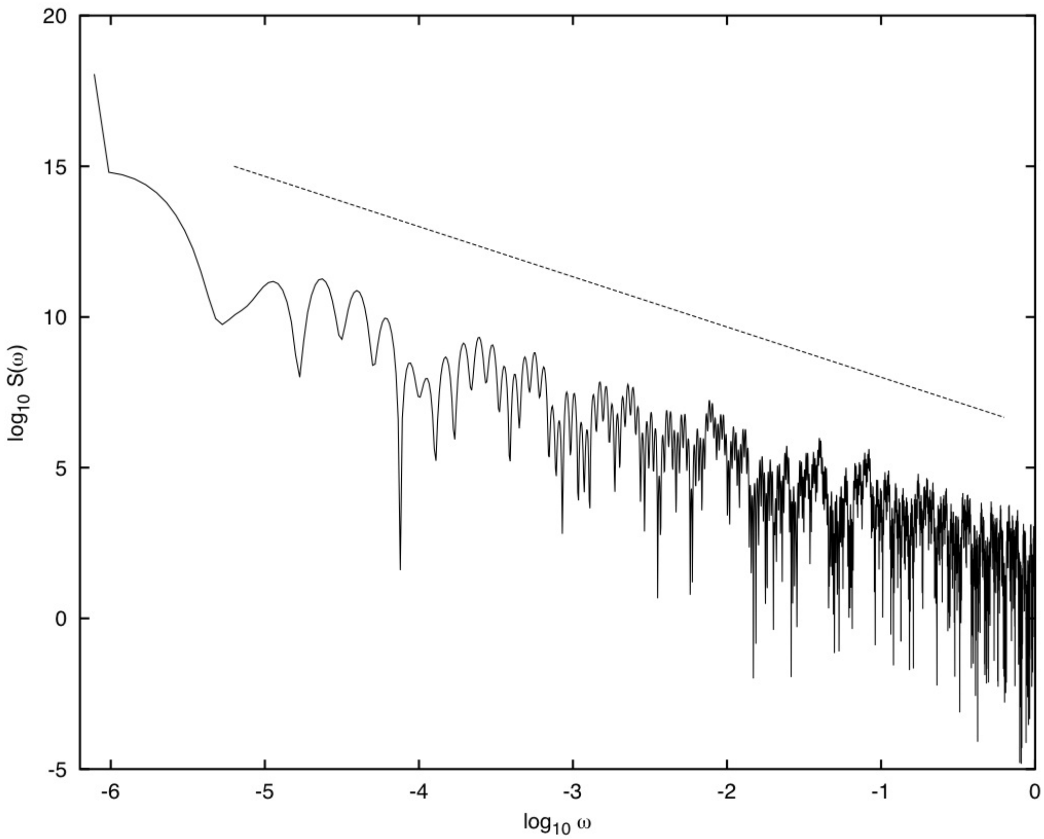

Figure 7: Plot of the power spectrum of the accumulating singularity function of *f(t)* from equation 11 and the dashed line showing the -5/3 spectral slope.

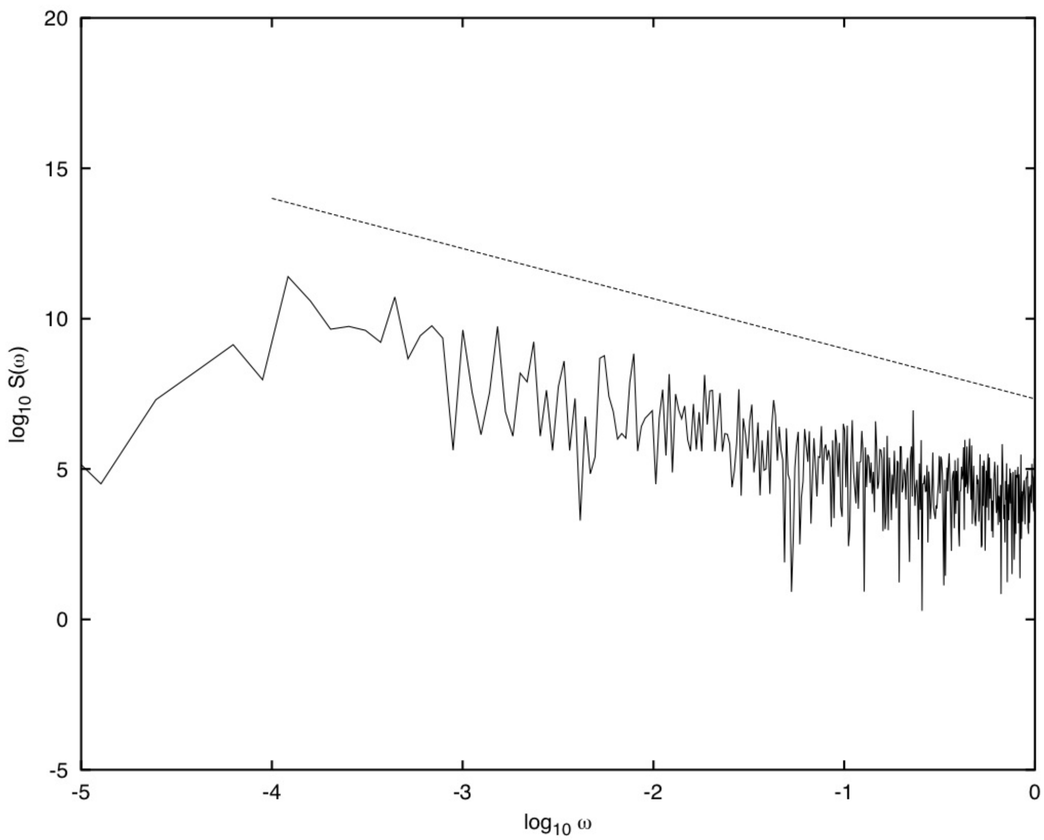

Figure 8: Plot of the power spectrum of the Weierstrass function of *f(t)* from equation 15 and the dashed line showing the -5/3 spectral slope.

The level crossing pdfs are shown in figures 9 and 10 respectively for the accumulating-singularity function in equation 11 and the Weierstrass function in equation 15. We see clearly that the level crossing pdfs are very different for these two functions even though the power spectra in both cases in figures 7 and 8 show the same -5/3 spectral scaling slope. This illustrates therefore the possibility of non-uniqueness of the level crossing statistics, i.e. that different level crossing pdfs can have the same power spectrum scaling slope.

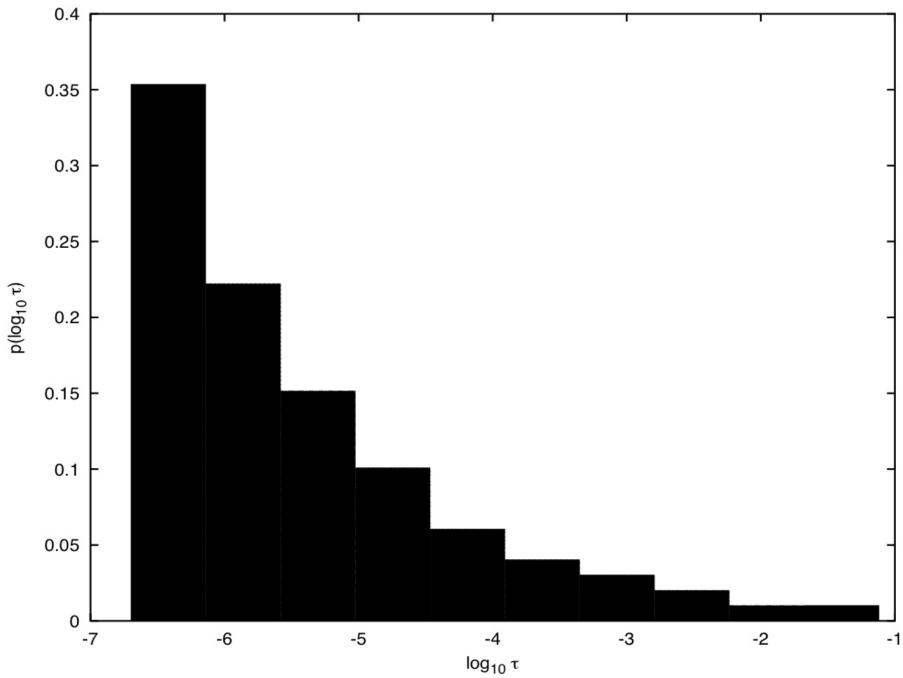

Figure 9: Plot of the level crossing pdf as a histogram for the accumulating singularity function of *f(t)* from equation 11.

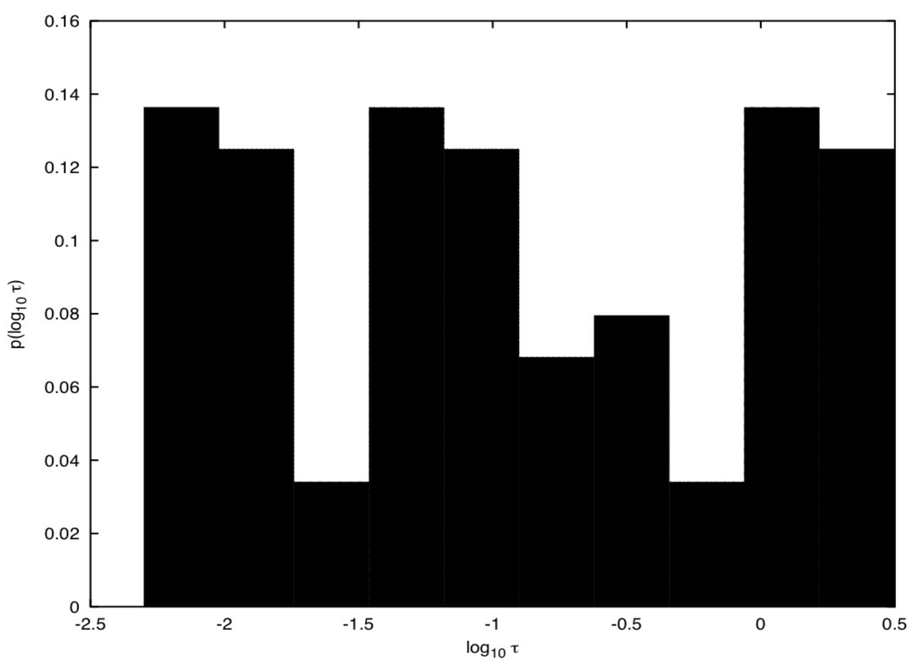

Figure 10: Plot of the level crossing pdf as a histogram for the Weierstrass function of *f(t)* from equation 15.

The above findings are especially important in the context of turbulent flows because there are various observations of different level crossing statistics (Catrakis 2000; Edwards & Hurst 2001; Kailasnath & Sreenivasan 1993; Mandelbrot 1975; Sreenivasan & Bershadskii 2006; Sreenivasan 1983; Vassilicos & Brasseur 1996; Yee et al. 1995) even though these flows all exhibit Kolmogorov spectral scaling with the same -5/3 spectral scaling slope in the inertial range of turbulent scales. Thus, one basic unanswered question has been whether or not there should be theoretically a universal level crossing pdf that corresponds to the Kolmogorov -5/3 spectral scaling. Our findings above illustrate mathematically that non-universality of the level crossing pdf is certainly a possibility, i.e. that we can expect different level crossing pdfs for the same spectral scaling. To what extent various turbulent flows exhibit different level crossing pdfs, is an important question fundamentally and practically that we hope will be explored with further theoretical considerations, computations, and observations of turbulence in future studies.

4. Conclusions

We have considered mathematical aspects of level crossings of fluctuating functions, especially with emphasis on spectral properties. Using a general approach for relating spectra and level crossing probability density functions, we have shown that a power spectrum can be non-uniquely related to different probability density functions of level crossing scales. As an illustrative example of this non-uniqueness, we have explored two different functions which have the same -5/3 power-law spectral slope and yet have different level crossing statistics. Our findings have implications in the context of turbulent flows because mathematically we see that there is certainly a possibility to have non-universality of the level crossing pdf, i.e. that there can be different level crossing pdfs for the same spectral scaling.

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