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NONORIENTABLE CONTACT STRUCTURES ON 3-MANIFOLDS

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# NONORIENTABLE CONTACT STRUCTURES ON 3-MANIFOLDS

DAVID CROMBECQUE

ABSTRACT. We exhibit the first examples of nonorientable tight contact structures, on the solid torus and the torus bundle over the circle, for which the pullback to the orientation double cover is overtwisted.

## 1. INTRODUCTION

Since Bennequin's work [1], it has been well-known that there is a dichotomy between tight and overtwisted contact structures in dimension three. Overtwisted contact structures are well-understood by the work of Y. Eliashberg [2], but tight contact structures are still far from being completely understood. In 1991, E. Giroux [5] introduced the notion of a convex surface, which enjoys two essential properties: flexibility and genericity. In 2000, K. Honda [7] introduced the notion of a bypass, which he combined with convex surface theory to classify tight contact structures on such manifolds as the solid torus, lens spaces and torus bundles which fiber over the circle [7, 8]. Similar results were also obtained by Giroux [6]. In most studies, contact structures are always assumed to be oriented. (Recall that a contact 3-manifold is always orientable but a contact structure on the 3-manifold does not need to be orientable.) It is often thought that if one has to deal with nonorientable contact structures, it suffices to pass to the orientation double cover. Although it is true that a nonorientable contact structure is tight if its pullback to the orientation double cover is tight, our motivation is to demonstrate that one cannot merely switch to the orientation double cover without loss of information when studying tightness. In this article, we systematically study the tightness of nonorientable contact structures and produce examples of 3-manifolds equipped with nonorientable tight contact structures for which the orientation double cover is overtwisted.

Recall that a large subclass of tight contact 3-manifolds arise as boundaries of symplectic 4-manifolds, and much of the early progress in 3-dimensional contact topology came from holomorphic techniques. Such contact structures obtained as boundaries of symplectic manifolds will necessarily be orientable. In our study of nonorientable contact structures, we therefore use a 3-dimensional cut-and-paste approach, where we cut contact manifolds along convex surfaces and check for tightness using dividing curves and bypasses. We then analyze nonorientable contact structures using the state transition method developed by Honda [8], on  $T^2 \times I$ ,  $S^1 \times D^2$ , and  $T^2$ -bundles over  $S^1$ .

## 2. CONVEX SURFACES

In this section we review convex surfaces and bypasses, with a particular emphasis on nonorientable contact structures defined in a neighborhood of oriented surfaces.

A *contact structure* on a 3-dimensional differentiable manifold  $M$  is a completely nonintegrable distribution  $\xi$  of planes in the tangent space  $TM$ . Locally, the planes can be described as the kernel of a one-form  $\alpha$ , where

$$\alpha \wedge d\alpha \neq 0.$$

This condition is independent of the choice of  $\alpha$  and the sign of  $\alpha \wedge d\alpha$  is independent of the sign of  $\alpha$ . In local coordinates  $\varphi_j : U_j \rightarrow \mathbb{R}^3$ , if  $\xi$  is given by  $\alpha_j$ , then there exist non-vanishing functions  $f_{ij} : U_i \cap U_j \rightarrow \mathbb{R}$  such that  $\alpha_j = f_{ij}\alpha_i$  on  $U_i \cap U_j$ . The volume forms  $\alpha_j \wedge d\alpha_j$  satisfy

$$\alpha_j \wedge d\alpha_j = f_{ij}^2 \alpha_i \wedge d\alpha_i.$$

It follows that  $\xi$  induces an orientation on  $M$ . Hence a contact 3-manifold is always orientable. However, even though  $d\alpha \neq 0$  on  $\xi$ ,  $d\alpha$  does depend on the sign of  $\alpha$ . Thus  $\xi$  may or may not be oriented. Now  $\xi$  is said to be *transversely oriented* if the local 1-forms  $\alpha_j$  can be chosen such that all transition functions  $f_{ij} > 0$ . Only then using a partition of unity, can one construct a globally defined 1-form  $\alpha$ .

A *contact vector field* on  $(M, \xi)$  is a vector field  $X \in \mathcal{X}(M)$  whose flow preserves  $\xi$ . A properly embedded surface  $S$  in  $(M, \xi)$  is called *convex* if there is a contact vector field  $X$  transverse to  $S$ . Since the contact vector field  $X$  induces a transverse orientation on  $S$ , a convex surface is always oriented. Also  $X$  allows us to find an  $I$ -invariant neighborhood  $S \times I \subset M$  of  $S$  where  $S = S \times \{0\}$ . A contact structure  $\xi$  is *vertically invariant* on  $S \times I$  if it is invariant by the flow of  $\frac{\partial}{\partial t}$  where  $t$  is the coordinate in the  $I$ -direction. It is locally defined by equations of the type  $\beta_j + u_j dt$  where  $\beta_j$  and  $u_j$  are respectively a 1-form and a function defined on an open subset  $U_j$  of  $S$  such that  $u_j d\beta_j + \beta_j \wedge du_j \neq 0$ .

Consider  $S$  a closed convex surface,  $X$  a contact vector field transverse to  $S$ . Define  $\Gamma_S = \{p \in S \text{ where } X(p) \in \xi(p)\}$ .  $\Gamma_S$  is called *the dividing set* of  $S$ . Note that the isotopy type of  $\Gamma_S$  does not depend on the choice of the contact vector field  $X$ . Notice that if the contact structure  $\xi$  is oriented in a neighborhood of a convex surface  $S$ , the dividing set  $\Gamma_S$  separates  $S$  into subsurfaces  $R_+ \sqcup R_-$  where  $R_+$  is the subsurface where the orientations of  $X$  and the normal orientation of  $\xi$  coincide, and  $R_-$  is the subsurface where they are opposite. However, in the case of a nonorientable contact structure,  $\Gamma_S$  does not divide  $S$  into positive and negative subsurfaces anymore.

Consider a properly embedded surface  $S$  in  $(M, \xi)$ . Except at the singular points  $p \in S$  where the tangent plane  $T_p S$  coincides with the contact plane  $\xi_p$ ,  $TS \cap \xi$  defines a line field. This line field determines a singular foliation  $\mathcal{F}_S$  called the *characteristic foliation* of  $S$ . If  $S$  is oriented and  $\xi$  is

transversely oriented, then  $\mathcal{F}_S$  will also be oriented. On a convex surface  $S$ , if  $\xi$  is transversally oriented, given a volume form  $\omega$  on  $S$ , the characteristic foliation  $\mathcal{F}_S$  is directed by a vector field  $Y$  which will induce an equation for  $\xi$ , namely  $i(Y)\omega + dt = 0$ . If  $\mathcal{F}$  is a singular foliation on a convex surface  $S$ , a properly embedded multi-curve  $\Gamma$  is said to *divide*  $\mathcal{F}$  if there exists some vertically invariant contact structure  $\xi$  on  $S \times I$  such that  $\mathcal{F}$  is the characteristic foliation  $\mathcal{F}_{S \times \{0\}}$  on  $S \times \{0\}$  and  $\Gamma$  is the dividing set for  $S \times \{0\}$ .

In order to analyze nonorientable contact structures, we will make use of Giroux's Flexibility Theorem. Although this theorem is most often used to study orientable contact structures, note that the theorem still applies in the nonorientable case

**Theorem 2.1** (Giroux's Flexibility Theorem). *Let  $S$  be a convex surface in  $(M, \xi)$ ,  $X$  a contact vector field transverse to  $S$  and  $\Gamma$  the dividing set of  $S$ . Let  $\mathcal{F}$  be a singular foliation on  $S$  such that  $\Gamma$  divides  $\mathcal{F}$ . Then there exists an isotopy  $\Phi_s$ ,  $s \in [0, 1]$  of  $S$  such that*

- (1)  $\Phi_0(S) = S$ ;
- (2)  $\Phi_1(\mathcal{F})$  is the characteristic foliation of  $\Phi_1(S)$ ;
- (3)  $\Phi_s(S)$  is transverse to  $X$  for any  $s \in [0, 1]$ ;
- (4) the dividing set associated to  $X$  on  $\Phi_s(S)$  is  $\Phi_s(\Gamma)$ .

The key to adapting the proof to the case of a nonorientable contact structure  $\xi$  is to realize that if  $S$  is a convex surface, then  $\xi$  will be transversely oriented on each connected component of  $S \setminus \Gamma_S$ , since the contact vector field is transverse to it. Giroux's proof applies to each component of  $S \setminus \Gamma_S$ .

Convex surfaces are generic. Any closed surface in a manifold equipped with a nonorientable contact structure can become convex after a  $C^\infty$ -small perturbation.

**Definition 2.2.** *A singular foliation  $\mathcal{F}$  on a closed surface  $S$  is said to be Morse-Smale if it satisfies the following conditions:*

- (1) the singularities and closed leaves of  $\mathcal{F}$  are hyperbolic;
- (2) the limit set of every trajectory is either a singularity or a closed leaf;
- (3) there are no saddle-saddle connections.

**Proposition 2.3** (Giroux). *Let  $S$  be a closed oriented surface in  $(M, \xi)$ . If the characteristic foliation  $\mathcal{F}$  is Morse-Smale, then  $S$  is convex.*

**Theorem 2.4.** *On a closed oriented surface, the characteristic foliation is generically Morse-Smale.*

*Proof.* We apply Peixoto's theorem [11] which asserts that the set of Morse-Smale vector fields on a closed orientable surface is open and dense in the space of all vector fields on  $S$  with the  $C^1$ -topology. Recall that the characteristic foliation is given locally by vector fields  $X_i$  such that

$i(X_i)\theta = \beta_i$  where  $\beta_i$  is a local contact form and  $\theta$  a local volume form on  $S$ . Now  $\mathcal{F}$  is not orientable since  $\xi$  is not. So we cannot find a global vector field  $X$  directing  $\mathcal{F}$ . Let us then define the space of line fields on  $S$ , denoted  $\mathcal{L}(S)$ , by  $\mathcal{L}(S) = C^\infty(S) \setminus \{-1, 1\}$  equipped with the induced  $C^1$ -topology. The characteristic foliation  $\mathcal{F}$  of  $S$  is then given a line field  $L$  of  $\mathcal{L}(S)$ .  $\square$

**Corollary 2.5.** *Let  $S$  be a closed surface embedded in a 3-manifold  $(M, \xi)$  where  $\xi$  is nonorientable. Then there exists a  $C^\infty$ -small isotopy of  $S$  so that it becomes a convex surface.*

Next we review another essential tool to our study of tightness. This is the notion of *bypass* and is due to K. Honda [7].

**Definition 2.6.** *Let  $\Sigma$  be a convex surface. A bypass is a half-disk  $D$  with  $\partial D = \alpha \cup \beta$  Legendrian for which the following hold:*

- (1)  $\alpha = \Sigma \cap D$ ;
- (2)  $\Gamma_\Sigma \cap \{p_1, p_2, p_3\}$ , where  $p_1, p_2, p_3$  are distinct points;
- (3)  $\alpha \cap \beta = \{p_1, p_3\}$ ;
- (4) for one orientation of  $D$ ,  $p_1$  and  $p_3$  are both elliptic singular points of  $D$ ,  $p_2$  is negative elliptic, and all the singular points along  $\beta$  are positive and alternate between elliptic and hyperbolic.

Suppose we cut along  $M$  along convex surface with Legendrian boundary. The following are ways in which a bypass can occur.

**Lemma 2.7.** *Let  $\Sigma = D^2$  be a convex surface with Legendrian boundary inside a tight contact manifold. If  $t(\partial\Sigma) < -1$ , then there exists a bypass along  $\partial\Sigma$ .*

**Proposition 2.8.** *(Imbalance Principle) Let  $\Sigma = S^1 \times [0, 1]$  be convex with Legendrian boundary inside a tight contact manifold. If  $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$ , then there exists a bypass along  $S^1 \times \{0\}$ .*

We remark that the technique of bypass attachment from [7] carries over to the nonorientable case without any modification.

**Definition 2.9.** *We define an abstract bypass move as follows*

- (1) *Start with a multicurve  $\Gamma$  on a closed or compact surface  $S$ , and an arc  $\delta$  which transversely intersects exactly three points of  $\Gamma$ , two of them at  $\partial\delta$ .*
- (2) *Modify  $\Gamma$  to  $\Gamma'$ , obtained as though there were an actual bypass and the dividing set were modified under an isotopy of  $S$ .*

*For an abstract bypass move, the physical presence of a bypass is not necessary.*

**Definition 2.10.** *A bypass attachment is called trivial if it does not change the configuration of the dividing set.*

We next describe the State Transition method due to K. Honda: [9]

Let  $M$  be a handlebody of genus  $g$  so that  $\Sigma = \partial M$  is convex and  $D_1, \dots, D_g$  be compressible disks so that  $M \setminus (D_1 \cup \dots \cup D_g) = B^3$ . Fix  $\Gamma_\Sigma$ . Suppose that  $|\partial D_i \cap \Gamma_\Sigma| \neq 0$  and  $\#(\partial D_i \cap \Gamma_\Sigma) = |\partial D_i \cap \Gamma_\Sigma|$ . We make  $\partial D_i$  Legendrian by applying Proposition 2.4.3 to  $\Sigma$  and then perturb  $D_i$  so that it becomes convex.

Let  $\mathcal{C}$  be the configuration space, namely the set of all possible configurations  $C = (\Gamma_{D_1}, \dots, \Gamma_{D_g})$  and introduce a directed graph  $G = (\mathcal{C}, \mathcal{T})$ , where the configuration space  $\mathcal{C}$  is the set of vertices and  $\mathcal{T} \subset \mathcal{C} \times \mathcal{C}$  is the set of directed edges, called *allowable state transitions*. We will write  $C \rightarrow C'$  for  $(C, C') \in \mathcal{T}$ .

A configuration  $C$  gives rise to  $\Gamma_{B^3}$  after edge-rounding.  $C$  is said to be *potentially allowable* if  $\Gamma_{\partial B^3} = S^1$ . We say a state transition is *allowable* and write  $C \rightarrow C'$  if

- (1)  $C$  is potentially allowable;
- (2)  $C'$  can be obtained from  $C$  via a single nontrivial abstract bypass attachment along some  $D_i$ .
- (3) Performing an abstract bypass move along a Legendrian arc on  $\partial B^3$  from the interior of  $B$  does not change  $\#\Gamma_{\partial B^3}$ .

$C \rightarrow C'$  implies  $C' \rightarrow C$ , unless  $C'$  is already not potentially allowable. A configuration  $C$  is *allowable* if every  $C' \in \mathcal{C}$  in the same connected component of  $G$  is potentially allowable. Denote  $\mathcal{C}_0$  the set of allowable  $C \in \mathcal{C}$ . On  $\mathcal{C}_0$  the graph is reflexive and we write  $\pi_0(\mathcal{C}_0)$  to mean the connected components of  $\mathcal{C}_0$ .

**Theorem 2.11.** *Let  $\mathcal{T}(M, \Gamma_\Sigma)$  be the set of isotopy classes of tight contact structures on  $M$ , relative to the boundary  $\Sigma$ . Then the map  $\psi: \mathcal{C}_0 \rightarrow \mathcal{T}(M, \Gamma_\Sigma)$  is surjective and factors through*

$$\pi_0(\mathcal{C}_0) \xrightarrow{\sim} \mathcal{T}(M, \Gamma_\Sigma).$$

**Corollary 2.12.** *Let  $[C] \in \pi_0(\mathcal{C}_0)$  be the connected component containing  $C$ . If  $[C]$  contains only one configuration, then the corresponding contact structure is universally tight.*

### 3. NONORIENTABLE CONTACT STRUCTURES ON $T^2 \times I$

In this section, we will analyze nonorientable contact structures on  $T^2 \times [0, 1]$ . We begin by considering a convex torus  $\Sigma$  in a manifold  $M$  equipped with a tight nonorientable contact structure  $\xi$ . We choose an oriented identification of  $\Sigma$  with  $\mathbb{R}^2/\mathbb{Z}^2$ , where the dividing set  $\Gamma_\Sigma$  consists of  $2n + 1$  parallel, homotopically essential curves of slope  $\infty$ .

**Proposition 3.1.** *Let  $\Sigma = T^2$  be a convex torus with dividing set  $\Gamma_\Sigma$  consisting of  $2n + 1$  parallel, homotopically essential curves of slope  $s(T^2) = \infty$ , and assume a bypass  $D$  is attached from the back along a Legendrian ruling of slope  $r$  with  $-1 < r < 0$ . Then there exists a*

neighborhood  $T^2 \times I$  of  $\Sigma \cup D$  with  $\partial(T^2 \times I) = T_1 - T_0$ , so that  $\Gamma_{T_0} = \Gamma_\Sigma$  and  $\Gamma_{T_1}$  will be as follows, depending on whether  $\#\Gamma_{T_0} = 1$  or  $\#\Gamma_{T_0} > 1$ :

- (1) If  $\#\Gamma_{T_0} > 1$ , then  $s(T_0) = s(T_1) = \infty$  but  $\#\Gamma_{T_1} = \#\Gamma_{T_0} - 2$ .
- (2) If  $\#\Gamma_{T_0} = 1$  and  $-\frac{1}{2} \leq r < 0$ , then  $\#\Gamma_{T_1} = 2$  and there is an overtwisted disk.
- (3) If  $\#\Gamma_{T_0} = 1$  and  $-1 < r < -\frac{1}{2}$ , then  $\#\Gamma_{T_1} = 1$  and  $s(T_1) = -\frac{3}{4}$ .

In 3.1, we have normalized the Legendrian rulings so that  $-1 < r < 0$ . Here  $\#\Gamma$  indicates the number of components of  $\Gamma$ .

*Proof.* This follows from the Bypass Attachment Lemma of [7]. Figure 1 illustrates the case  $\#\Gamma_{T_0} > 1$ . Notice that the number of components of the dividing set is reduced by two.

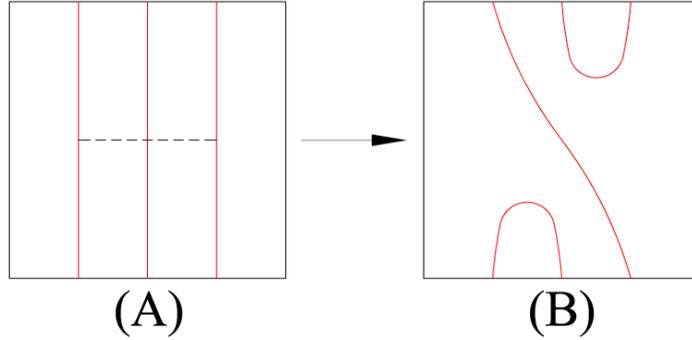


FIGURE 1. Bypass attachment for  $\#\Gamma > 1$ . The sides of the torus are identified and so are the top and the bottom. The dotted line represents the bypass arc of attachment.

In the case where  $\Gamma_{T_0} = 1$ , there are two possibilities depending on the slope of the Legendrian arc along which the bypass is attached. If the bypass is attached on a ruling curve of slope  $-\frac{1}{2} \leq r < 0$ , then the dividing set will have two components and will contain a homotopically trivial curve as in Figure 2. This bypass leads to an overtwisted disk and therefore cannot exist inside a tight contact manifold. Figure 3 illustrates the remaining case.

□

We can reinterpret the slope conditions of Cases 2 and 3 above by considering the Farey tessellation on the hyperbolic unit disk  $\mathbb{H}^2 = \{(x, y) | x^2 + y^2 \leq 1\}$ . In this tessellation, the points on the unit circle are identified with  $\mathbb{Q} \cup \{\infty\}$  so that  $(1, 0) \in \mathbb{H}^2$  is identified with  $\frac{0}{1}$ ,  $(-1, 0)$  is identified with  $\frac{-1}{0}$  and for each  $\infty > \frac{p}{q} > 0$  ( $p, q$  relatively prime) and  $\infty > \frac{p'}{q'} > 0$  ( $p', q'$  relatively prime) are such that  $(p, q), (p', q')$  form a

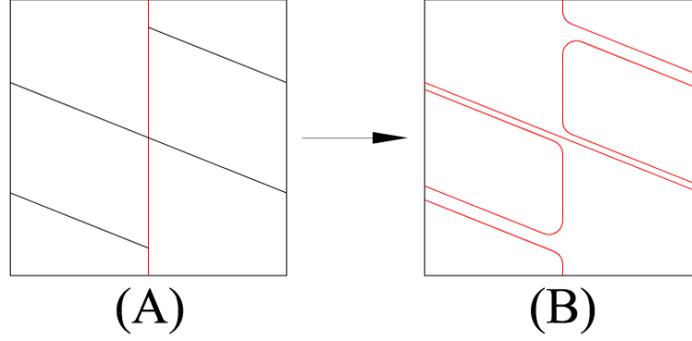


FIGURE 2. Bypass attachment for  $\#\Gamma = 1$  and  $-\frac{1}{2} \leq r < 0$ .

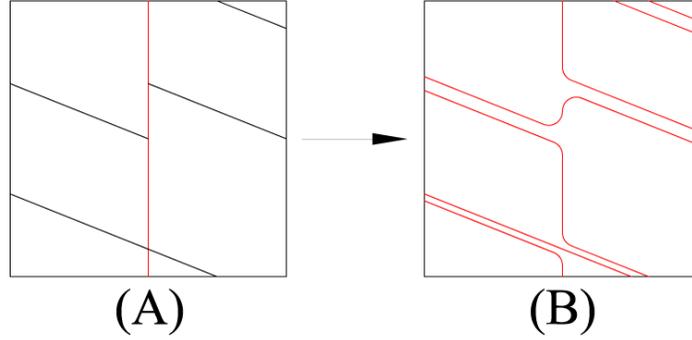


FIGURE 3. Bypass attachment for  $\#\Gamma = 1$  and  $-1 < r < -\frac{1}{2}$ .

$\mathbb{Z}$ -basis of  $\mathbb{Z}^2$ , then the point labeled  $\frac{p+p'}{q+q'}$  sits on  $S^1$  halfway between  $\frac{p}{q}$  and  $\frac{p'}{q'}$  on the arc for which  $y$  is always positive. Refer to Figure 4

**Proposition 3.2.** *Let  $\Sigma = T^2$  be a convex surface with  $\#\Gamma_\Sigma = 1$  and slope  $s = -\frac{p}{q}$ . Let us denote  $r = -\frac{p'}{q'}$  with  $0 < p' < p$  and  $0 < q' < q$  ( $p', q'$  relatively prime) such that  $-pq' + p'q = -1$ . Assume a bypass is attached from the back along a curve of slope  $m$ .*

- (1) *On the Farey tessellation, if  $m$  is on the arc from  $-\frac{p-2p'}{q-2q'}$  to  $r$  going clockwise, then the resulting convex surface  $\Sigma'$  will have  $\#\Gamma_{\Sigma'} = 1$  and the new slope  $s'$  will be obtained as follows: Take the arc from  $s$  to  $r$  going clockwise. On this arc, let  $s'$  be the fourth point sharing an edge with  $r$ .*

- (2) If  $m$  is on the arc from  $-\frac{p-p'}{q-q'}$  to  $-\frac{p-2p'}{q-2q'}$  going clockwise, then the resulting convex surface  $\Sigma'$  will have  $\#\Gamma_{\Sigma'} = 2$ .

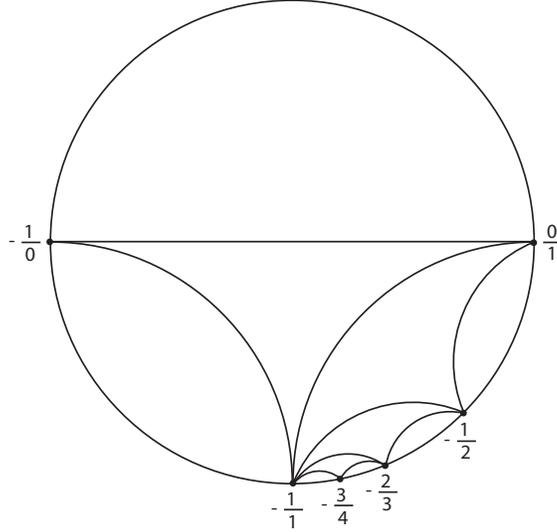


FIGURE 4. Interpretation of Case (3) on the Farey tessellation.

Figure 4 is an illustration of the proposition when  $s = -\infty, r = -1$ ,  $m$  is anywhere on the arc  $(-\frac{1}{2}, -1)$  and  $s' = -\frac{3}{4}$ .

*Proof.* Note that the bypass attachment performed in Case 3 above actually creates four twists around the Legendrian curve of slope  $-1$ . Each twist corresponds to a hop in the Farey tessellation to the next point sharing an edge with  $-1$ . Thus in the Farey tessellation, moving clockwise along the arc from  $-\infty$  to  $-1$ , we are doing a sequence of four hops on the points having an edge with  $-1$ . We can now transform the previous situation through  $SL(2, \mathbb{Z})$  to the case where  $s = -\frac{p}{q}$  and  $r = -\frac{p'}{q'}$  as described in Proposition 3.2.  $\square$

**Corollary 3.3.** *Let  $\Sigma = T^2$  be a convex torus with  $\#\Gamma = 1$  and slope  $s = -\frac{p}{q}$ , ( $p$  and  $q$  relatively prime integers greater than 1). If a bypass  $D$  is attached from the back along a curve of slope  $m$  contained in the arc from  $-\frac{p-2p'}{q-2q'}$  to  $r$  going counterclockwise where  $r = -\frac{p'}{q'}$  with  $0 < p' < p$  and  $0 < q' < q$  ( $p', q'$  relatively prime) such that  $-pq' + p'q = -1$ , then the new dividing set will have one component with slope  $s' = -\frac{p-4p'}{q-4q'}$ .*

*Proof.* This follows from the definition of  $s'$  in the above proposition and the fact that if you have two points  $-\frac{p}{q}$  and  $-\frac{p'}{q'}$  sharing an edge, then the

next point going clockwise on the arc going from  $-\frac{p}{q}$  to  $-\frac{p'}{q'}$  sharing an edge with  $-\frac{p'}{q'}$  is given by  $-\frac{p-p'}{q-q'}$ . Repeating the operation four times results in the desired slope.  $\square$

We are now in a position to analyze nonorientable tight contact structures on  $T^2 \times I$ . In the following propositions, we consider  $T^2 \times I$  with the following boundary conditions, which we call (A):

$$\#\Gamma_0 = \#\Gamma_1 = 1; s_0 = -\frac{3}{4}, s_1 = -\infty.$$

**Proposition 3.4.** *There exists a tight contact structure  $(T^2 \times I, \xi)$  satisfying (A), and which arises from a single bypass attachment as in Theorem 3.1.*

*Proof.* Consider the convex torus  $T$  with dividing curve of slope  $\infty$  and arc of bypass attachment along a ruling curve of slope  $-1 < r < -\frac{1}{2}$  as in Theorem 3.1. To prove the tightness of the contact structure on  $T \times I$  given by the single bypass attachment, we pass to the orientation double cover and prove tightness there. Let  $T' \rightarrow T$  be the orientation double cover, obtained by expanding in the  $(1, 0)$ -direction. The convex torus  $T'$  has dividing set  $\Gamma_{T'}$  satisfying  $\#\Gamma_{T'} = 2$  and slope  $= -\infty$ . Moreover, the single bypass attached to  $T$  becomes two bypasses attached to  $T'$ . If we attach the two bypasses in order, we obtain the following sequence of slopes  $(-\infty, -2, -\frac{3}{2})$ .

We now appeal to a result from the classification of orientable tight contact structures on  $T^2 \times I$  [7].

**Lemma 3.5.** *Let  $(T^2 \times [0, 1], \xi)$  be a contact manifold which admits a factorization  $T^2 \times I = \cup_{i=0}^{k-1} N_i$ , where each  $N_i = T^2 \times [\frac{i}{k}, \frac{i+1}{k}]$  is a basic slice, and  $s_0 > s_{\frac{i}{k}} > \dots > s_1$  is obtained by taking the shortest counterclockwise sequence of hops along edges from  $s_1$  to  $s_0$  on  $\partial\mathbb{H}^2$ . Then  $(T^2 \times [0, 1], \xi)$  is tight.*

Applying the above lemma, we see that the contact structure on  $T' \times I$  is tight. This completes the proof of Proposition 3.4.  $\square$

The following proposition will be useful later, when applying state transitions. It is rather surprising, especially when compared to the orientable case — cf. the remark after the proposition.

**Proposition 3.6.** *Inside  $(T^2 \times I, \xi)$  there is no convex torus  $T$  parallel to  $T_0$  or  $(T_1)$  whose dividing set has slope  $\neq -\infty, -\frac{3}{4}$ .*

*Proof.* Assume the existence of a torus  $T'$  parallel to  $T_1$  with slope  $s$  in  $(-\infty, -1)$ . The orientation double cover is given by  $T^2 \times [0, 1]$  with the following convex boundary conditions:  $\#\Gamma_0 = \#\Gamma_1 = 2, s_1 = \infty, s_0 = -\frac{3}{2}$ . Using the Imbalance Principle (see [7]), there exists a bypass along  $T_1$ . Factoring  $T^2 \times I$ , we get the following sequence of slopes:  $-\frac{3}{2}, -2, \infty$ . But

$T'$  is now a convex torus of slope  $2s \in (-\infty, -2)$ . Now this point  $2s$  is not on the shortest path from  $-\frac{3}{2}$  to  $\infty$ . This leads to a contradiction.  $\square$

**Remark 3.7.** *Observe the difference with a basic slice  $(T^2 \times I, \xi)$  in the case when  $\xi$  is oriented. In that case, for any rational slope  $s$ , there is a convex torus parallel to the boundary whose dividing set attains slope  $s$ .*

#### 4. NONORIENTABLE CONTACT STRUCTURES ON THE SOLID TORUS

In this section, we give the first examples of nonorientable tight contact structures which become overtwisted when pulled back to the oriented double cover. We fix an identification of  $T^2 = \partial(S^1 \times D^2)$  with  $\mathbb{R}^2/\mathbb{Z}^2$ , where  $(1, 0)^T$  is the meridian circle (slope is 0) and  $(0, 1)^T$  the longitude (slope is  $\infty$ ).

**Theorem 4.1.** *Consider the solid torus  $S^1 \times D^2$  with convex boundary  $\Sigma = T^2$  and dividing set  $\Gamma_\Sigma$  consisting of one curve of slope  $s(\Gamma) = -\frac{6}{5}$ . Then there exists a unique nonorientable tight contact structure  $\xi$ . Its orientation double cover is overtwisted.*

*Proof.* First let us find tight contact structures on our solid torus. To achieve this goal, we will use the state transition method (see [9] ) and Eliashberg's classification theorem for tight contact structures on the 3-ball  $B^3$ . Let  $D$  be a compressing disk for  $S^1 \times D^2$ . Applying the Legendrian realization principle to  $\Sigma$ , we make  $\partial D$  Legendrian and then perturb  $D$  so that it becomes convex with Legendrian boundary. We then cut  $S^1 \times D^2$  along  $D$  to obtain a 3-ball  $B^3$ . Let  $\mathcal{C}$  be the set of potentially allowable configurations. We claim that  $\mathcal{C}$  consists of  $\{(1), (2), (3)\}$ , as given in Figure 5, which represents the different configurations of the dividing curve on  $D$  after cutting  $S^1 \times D^2$ .

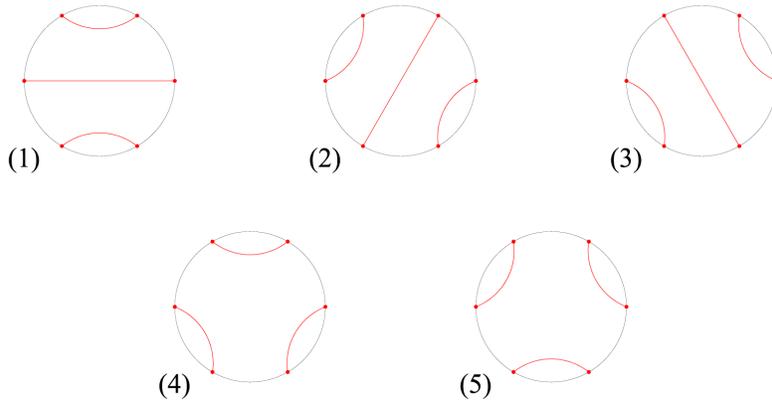


FIGURE 5. Set Of all configurations.

Observe that the configurations where  $\Gamma_D$  is  $\partial$ -parallel are disallowed since they lead to  $\#\Gamma_{\partial B^3} > 1$  after edge-rounding whereas configurations  $\{(1), (2), (3)\}$  lead to  $\#\Gamma_{\partial B^3} = 1$ . One can easily compute the state transitions  $(1) \leftrightarrow (2) \leftrightarrow (3) \leftrightarrow (1)$  where each state transition is given by a single bypass move which keeps  $\#\Gamma_{B^3} = 1$ . We describe the transition from (1), to (2) in Figure 6. Configuration (2) is obtained from Configuration (1) by attaching an abstract bypass along  $D$ . Similarly, Configuration (3) is obtained from Configuration (2) and Configuration (1) is obtained from Configuration (3). The dotted lines represent the arc of attachment of the abstract bypass.

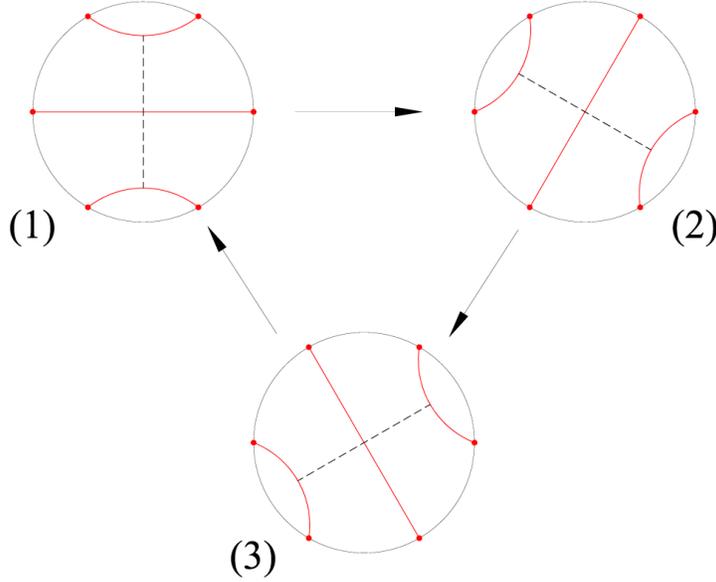


FIGURE 6. State transition.

Thus, by Theorem 2.11 we have found a nonorientable tight contact structure on  $S^1 \times D^2$  with  $\#\Gamma_{T^2} = 1$ . This contact structure is unique up to isotopy relative to the boundary, since the three states are the only allowable ones and are connected to each other. Now the oriented double cover of  $(S^1 \times D^2, \xi)$  is a solid torus  $(S^1 \times D^2, \xi')$  for which the dividing curve  $\Gamma'$  has 2 components of slope  $s' = -\frac{3}{5}$ . The configurations (1), (2) and (3) will lead to  $\#\Gamma'_{\partial B^3} > 1$ , implying the overtwistedness of  $\xi$ .  $\square$

**Remark 4.2.** *In the oriented case (see [7]), on the solid torus  $S^1 \times D^2$  with convex boundary  $\Sigma = T^2$  and dividing set  $\Gamma_\Sigma$  consisting of two parallel curves of slope  $s(\Gamma) = -\frac{6}{5}$ , there are two tight contact structures up to isotopy fixing the boundary, and they are universally tight.*

Let  $(S^1 \times D^2, \xi)$  be the solid torus with convex boundary  $T^2$  such that  $\#\Gamma_{T^2} = 1$  and slope  $s(\Gamma) = -\frac{p}{q}$  satisfying  $-\infty < -\frac{p}{q} \leq -1$ . Recall that  $(p, q)$  are relatively prime and  $p$  is even. Our hope is to be able to assess the tightness (or not) of  $(S^1 \times D^2, \xi)$ , given the slope  $-\frac{p}{q}$ . Two other cases are studied next: first when  $-\frac{p}{q} = -\frac{4n}{4n-1}$ , for  $n \geq 1$  then when  $-\frac{p}{q} = -2$ .

**Proposition 4.3.** *Consider the solid torus  $(S^1 \times D^2, \xi)$  with convex boundary  $T^2$  such that  $\#\Gamma_{T^2} = 1$  and slope  $s(\Gamma) = -\frac{4n}{4n-1}$ , for  $n \geq 1$ . There is no tight contact structure on  $S^1 \times D^2$ .*

*Proof.* First we show this is true for  $n = 1$  (i.e., the slope  $s(\Gamma) = -\frac{4}{3}$ ). Using again the state transition method, we see that the only possible configurations for  $\Gamma_D$  are  $\partial$ -parallel (see Figure 7) and lead to  $\Gamma_{\partial B^3} > 1$ . Thus, the corresponding solid torus is overtwisted.

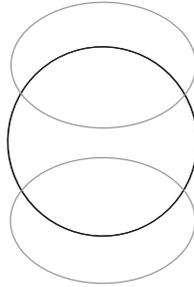


FIGURE 7. Configuration for curve of slope  $-\frac{4}{3}$ .

We can then prove the statement by induction on  $n$ . Assume there is no tight contact structure on the solid torus  $(S^1 \times D^2, \xi)$  with convex boundary  $T^2$  such that  $\#\Gamma_{T^2} = 1$  and slope  $s(\Gamma) = -\frac{4n}{4n-1}$ . Consider then the solid torus  $(S^1 \times D^2, \xi)$  with boundary slope  $-\frac{4(n+1)}{4(n+1)-1}$ . Let  $D$  be a meridian disk with Legendrian boundary and  $tb(\partial D) = -4(n+1)$ . Using the Imbalance Principle, there exists a bypass along  $\partial D$ . Attaching the bypass to the  $\partial(S^1 \times D^2)$  and applying Corollary 3.1.3, we know that attaching a bypass to the torus of slope  $-\frac{4(n+1)}{4(n+1)-1}$  will lead to a new slope  $-\frac{4n}{4n-1}$ . We can then prove the theorem by induction on  $n$ .  $\square$

**Remark 4.4.** *Again, in the oriented case, on the solid torus  $S^1 \times D^2$  with convex boundary  $\Sigma = T^2$  and dividing set  $\Gamma_\Sigma$  which consists of two parallel curves of slope  $s(\Gamma) = -\frac{4n}{4n-1}$ , there are two tight contact structures up to isotopy fixing the boundary.*

**Example 4.5.** *Consider the solid torus  $S^1 \times D^2$  with convex boundary  $\Sigma = T^2$  and dividing set  $\Gamma_\Sigma$  which consists of one curve of slope*

$s(\Gamma) = -2$ . *There exists a unique nonorientable tight contact structure. Its orientation double cover is also tight.*

*Proof.* In this case, there is only one configuration since there is only one arc, it is potentially allowable. Thus, there is a unique nonorientable tight contact structure on  $S^1 \times D^2$ . Since there is only one configuration, the contact structure is actually universally tight and thus its oriented double cover, namely  $S^1 \times D^2$  with convex boundary  $\Sigma = T^2$  and dividing set  $\Gamma_\Sigma$  which consists of 2 parallel curves of slope  $s(\Gamma) = -1$ , is also tight.  $\square$

Next we exhibit an example of a nonorientable tight contact structure on a closed manifold for which the pullback to the orientation double cover is overtwisted.

### 5. EXAMPLE ON THE TORUS BUNDLE OVER $S^1$

A  $T^2$ -bundle  $M$  over  $S^1$  can be viewed as  $T^2 \times [0, 1]$  with coordinates  $(x, t)$ , whose boundary components are glued via the monodromy map  $A : T^2 \times \{1\} \rightarrow T^2 \times \{0\}$ , where  $(x, 1) \mapsto (Ax, 0)$ . The  $T^2$ -bundle isomorphism type only depends on the conjugacy class  $[A]$  in  $SL(2, \mathbb{Z})$ .

**Theorem 5.1.** *Let  $M$  be the  $T^2$ -bundle over  $S^1$  with monodromy map  $A$  given by  $\begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$ . Then there exists a nonorientable tight contact structure on  $M$  such that its orientation double cover is overtwisted.*

*Proof.* In this proof we will apply the state traversal method [8] which can be described as follows: decompose your contact manifold  $(M, \xi)$  as  $M = M_1 \cup \dots \cup M_k$  where each  $M_i$  is irreducible, each boundary component of  $\partial M_i$  is incompressible and the union of all the boundaries is convex. Assume that we can determine whether  $\xi|_{M_i}$  is tight. A state is a collection  $\{(M_i, \xi_i) | i = 1, \dots, k\}$  such that the contact structures  $\xi_i$  glue to form a contact structure isotopic to  $\xi$ . The state is called a *tight state* if each of the  $\xi_i$  is tight.  $\xi$  will be tight if, starting with a tight state, we can show that all states that can be obtained from from this one are also tight. Let  $M$  be as described in Theorem 5.0.7. The initial state is obtained by cutting  $M$  along the fiber torus  $T^2 \times \{1\}$ , taken to be convex and with boundary conditions  $\#\Gamma_{T_1} = 1$  and slope  $s_1 = -\infty$ . Hence the initial state consists of  $T^2 \times I$  is with  $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 1$  and respective slopes  $s_0 = -\frac{3}{4}$ ,  $s_1 = -\infty$ , namely a basic slice which is tight according to Proposition 3.2.2. The initial state is thus tight. By Proposition 3.2.4, this is the only possible state. We use the monodromy map  $A$  to obtain  $M$ . Hence by the state traversal method, we obtain a nonorientable tight contact structure on  $M$ .

The oriented double cover of  $(M, \xi)$  is given by  $T^2 \times I$  with monodromy map  $A' = \begin{pmatrix} 5 & 2 \\ -8 & -3 \end{pmatrix}$  where  $\#\Gamma_{T'_0} = \#\Gamma_{T'_1} = 2$ ,  $s'_0 = -\frac{3}{2}$ ,  $s'_1 = -\infty$ .

Factoring  $T^2 \times [0, 1]$  into  $(T^2 \times [0, \frac{1}{2}]) \cup (T^2 \times [\frac{1}{2}, 1])$ , we get the following

sequence of boundary slopes  $-\infty, -2, -\frac{3}{2}$ . Moving  $(T^2 \times [\frac{1}{2}, 1])$  to the back via  $A$ , leads to the sequence  $(-2, -\frac{3}{2}, -2)$ . Using Lemma 3.2.3 from the classification of oriented tight contact structures on  $T^2 \times I$ , the contact structure on  $T^2 \times [-\frac{1}{2}, \frac{1}{2}]$  is overtwisted. Hence,  $(M, \xi)$  is overtwisted. (Referring to the classification of oriented contact structures on  $T^2$ -bundle over  $S^1$ ,  $A'$  is in the case where the rotation is large, namely, at least  $\pi$ ).  $\square$

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