PERIODIC EIGENFUNCTIONS OF THE FOURIER TRANSFORM OPERATOR

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Abstract. Let the generalized function (tempered distribution) f on $\mathbb{R}$ be a p-periodic eigenfunction of the Fourier transform operator $\mathcal{F}$, i.e.,

$$f(x+p) = f(x), \qquad \mathcal{F}f = \lambda f$$

for some $\lambda \in \mathbb{C}$. We show that

$$\lambda = 1, -i, -1, \text{or } +i,$$

that

$$p = \sqrt{N}$$

for some $N = 1, 2, \ldots$, and that f has the representation

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} \gamma[n]\delta\left(x - \frac{n}{p} - mp\right)$$

where δ is the Dirac functional and γ is an eigenfunction of the discrete Fourier transform operator $\mathcal{F}_N$ with

$$(\mathcal{F}_N\, \gamma)[k] = \frac{1}{N}\sum_{n=0}^{N-1} \gamma[n] e^{-2\pi i k n/N} = \frac{\lambda}{\sqrt{N}}\gamma[k], \qquad k = 0, 1, \ldots, N-1.$$

We generalize this result to p_1, p_2-periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^2$ and to p_1, p_2, p_3-periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^3$. We present numerous illustrations of such periodic eigenfunctions of $\mathcal{F}$ that are defined on the plane $\mathbb{R}^2$, and show how these can be used to create interesting quasiperiodic eigenfunctions of $\mathcal{F}$ having m-fold rotational symmetry within this context.

1. Introduction

In this paper we will study certain generalizations of the Dirac comb (or Ш functional, see [2, p. 117])

$$\text{Ш}(x) := \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{1}$$

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where δ is the Dirac functional. We work within the context of the Schwartz theory of distributions [10] as developed in [13], [8, Chapter 4, Chapter 8], [3, Chapter 7], [2, Chapter 3]. For purposes of manipulation we use "function" notation for δ, Ш and related functionals. More precise functional notation, e.g.,

$$\delta\{\phi\} := \phi(0), \qquad \phi \in \mathcal{S}(\mathbb{R})$$

$$Ш\{\phi\} := \sum_{n=-\infty}^{\infty} \phi(n), \qquad \phi \in \mathcal{S}(\mathbb{R})$$

is used where appropriate. Here

$$\mathcal{S}(\mathbb{R}) := \{\phi \in C^{\infty}(\mathbb{R}) : \sup|x^m \phi^{(n)}(x)| < \infty,\ m, n = 0, 1, 2, \ldots\} \tag{2}$$

is the linear space of univariate Schwartz functions. Various useful proprieties of δ and Ш are developed in [13], [8, Chapter 4, Chapter 8], [3, Chapter 7], [2, Chapter 3]. For example,

$$\delta\left(\frac{x}{a}\right) = |a|\delta(x), \qquad a > 0 \quad \text{or} \quad a < 0. \tag{3}$$

The Fourier transform of the Dirac comb can be obtained by using the Poisson sum formula

$$\sum_{n=-\infty}^{\infty} \phi(n) = \sum_{k=-\infty}^{\infty} \phi^{\wedge}(k), \qquad \phi \in \mathcal{S}(\mathbb{R})$$

where we use the superscript $\wedge$ for the Fourier transform

$$\phi^{\wedge}(s) := \int_{-\infty}^{\infty} e^{-2\pi i s x} \phi(x) dx.$$

Thus

$$\begin{aligned}
Ш^{\wedge}\{\phi\} &:= \int_{-\infty}^{\infty} Ш^{\wedge}(x)\phi(x)dx && \text{Integral notation for functional} \\
&:= \int_{-\infty}^{\infty} Ш(s)\phi^{\wedge}(s)ds, && \text{Definition of Fourier transform} \\
&=: \sum_{k=-\infty}^{\infty} \phi^{\wedge}(k), && \text{Action of Ш} \\
&= \sum_{n=-\infty}^{\infty} \phi(n), && \text{Poisson sum formula} \\
&=: Ш\{\phi\}, \quad \phi \in \mathcal{S}(\mathbb{R}), && \text{Action of Ш.}
\end{aligned}$$

We assume the knowledge of the remaining properties of the δ functional such as, the sifting property

$$\int_{-\infty}^{\infty} \delta(x-b)\phi(x)dx = \phi(b), \qquad b \in \mathbb{R},\ \phi \in \mathcal{S}(\mathbb{R}) \tag{4}$$

where the "integral" refers to the action of the functional $\delta(x-b)$, and the identity

$$g(x)\delta(x-a) = g(a)\delta(x-a) \tag{5}$$

(which holds when, $g, g^{'}, g^{''}, \ldots$ are all continuous functions of slow growth at $\pm\infty$).

The Ш functional is used in the study of sampling, periodization, etc., see [8, Chapter 5], [3, Chapters 7–9], [2, Chapter 7]. We will illustrate this process using a notation that can be generalized to an n-dimensional setting. Let $a_1 \in \mathbb{R}$ with $a_1 \neq 0$, and let $A_1 := \dfrac{1}{a_1}$. We define the lattice

$$\mathcal{L}_{a_1} := \{na_1 : n \in \mathbb{Z}\}$$

and the corresponding a_1-periodic Dirac comb

$$\operatorname{grid}_{a_1}(x) := \sum_{a \in \mathcal{L}_{a_1}} \delta(x-a). \tag{6}$$

We use (3) and (1) to write

$$\begin{aligned} \operatorname{grid}_{a_1}(x) &:= \sum_{n=-\infty}^{\infty} \delta(x - na_1) \\ &= \frac{1}{|a_1|} \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{a_1} - n\right) \\ &= \frac{1}{|a_1|} Ш\left(\frac{x}{a_1}\right), \end{aligned}$$

and thereby find

$$\begin{aligned} \operatorname{grid}^{\wedge}_{a_1}(s) &= Ш(a_1 s) \\ &= \sum_{k=-\infty}^{\infty} \delta(a_1 s - k) \\ &= \frac{1}{|a_1|} Ш\left(s - \frac{k}{a_1}\right) \\ &= |A_1| \operatorname{grid}_{A_1}(s). \end{aligned} \tag{7}$$

Let g be any univariate distribution with compact support. We can periodize g by writing

$$\begin{aligned} f(x) :&= \sum_{n=-\infty}^{\infty} g(x - na_1) \\ &= \sum_{n=-\infty}^{\infty} \delta(x - na_1) * g(x) \\ &= \operatorname{grid}_{a_1}(x) * g(x) \ , \end{aligned} \tag{8}$$

where $*$ represents the convolution product. We then use (8),(7) and (5) as we write

$$\begin{aligned} f^\wedge(s) &= |A_1| \operatorname{grid}_{A_1}(s)\ g^\wedge(s) \\ &= \sum_{k=-\infty}^{\infty} |A_1|\ \delta(s - kA_1)\ g^\wedge(s) \\ &= \sum_{k=-\infty}^{\infty} |A_1|\ g^\wedge(kA_1)\delta(s - kA_1)\ , \end{aligned}$$

and thereby obtain the weakly convergent Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} |A_1|\ g^\wedge(kA_1)e^{2\pi ikA_1x}. \tag{9}$$

We observe that $\operatorname{grid}_{a_1}$ has support at the points $na_1,\ n = 0, \pm1, \pm2, \ldots$ of the lattice $\mathcal{L}_{a_1}$, while the Fourier transform $|A_1| \operatorname{grid}_{A_1}$ has support at the points $\frac{n}{a_1},\ n = 0, \pm1, \pm2, \ldots$ of the lattice $\mathcal{L}_{A_1}$. It follows that

$$\operatorname{grid}^\wedge_{a_1} = \operatorname{grid}_{a_1}$$

if and only if

$$a_1 = \pm1,$$

i.e., if and only if

$$\operatorname{grid}_{a_1} = \text{III}. \tag{10}$$

Let $\mathcal{F}$ be the Fourier transform operator on the space of tempered distributions. It is well known [2], [3], [8], that $\mathcal{F}$ is linear and that

$$\mathcal{F}^4 = \mathcal{I}, \tag{11}$$

where $\mathcal{I}$ denotes the identity operator on the space of tempered distributions. We are interested in tempered distributions f such that

$$\mathcal{F}f = \lambda f, \tag{12}$$

where λ is a scalar. Any distribution f that satisfies (12), and that we will call eigenfunction of $\mathcal{F}$, must also satisfy the following equation

$$\mathcal{F}^n f = \lambda^n f \qquad \text{for any positive integer } n, \tag{13}$$

due to the linearity of the operator $\mathcal{F}$. When $n = 4$, then

$$\begin{aligned} \mathcal{F}^4 f &= \lambda^4 f \\ \mathcal{I} f &= \lambda^4 f \\ f &= \lambda^4 f. \end{aligned}$$

Thus the eigenvalues of the operator $\mathcal{F}$ are $1, -1, i, -i$.

2. Preliminaries

We would like to characterize all periodic eigenfunctions f of the Fourier transform operator $\mathcal{F}$, i.e.,

$$\mathcal{F} f = \lambda f, \qquad f \neq 0,$$

within the context of 1,2,3 dimensions. Such eigenfunctions are really tempered distributions, but we will continue to use "function" nomenclature.

2.1. Periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}$.

2.1.1. *Eigenfunctions of $\mathcal{F}_N$.* We first consider the periodic eigenfunctions of the discrete Fourier transform operator $\mathcal{F}_N$ since, as we will see later, they can be used to construct all periodic eigenfunctions of the Fourier transform operator $\mathcal{F}$.

Definition 2.1. Let $N = 1, 2, \ldots$. The matrix

$$\mathcal{F}_N := \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2N-2} \\ \vdots & \vdots & & & \vdots \\ 1 & \omega^{N-1} & \omega^{2N-2} & \ldots & \omega^{(N-1)(N-1)} \end{bmatrix}, \qquad \omega := e^{-2\pi i/N}$$

is said to be the discrete Fourier transform operator. In component form we have

$$(\mathcal{F}_N f)[k] := \frac{1}{N} \sum_{n=0}^{N-1} e^{-2\pi i k n/N} f[n], \qquad k = 0, 1, 2, \ldots, N-1. \tag{14}$$

We often use (14) in a context where both f and $\mathcal{F}_N f$ are N-periodic functions on $\mathbb{Z}$, see [3, Chapters 1,4,6], and we say that $f, \mathcal{F} f$ are functions on $\mathbb{P}_N$. In such cases, the summation can be taken over any N consecutive integers, and we allow k to take all values from $\mathbb{Z}$.

It is easy to verify the operator identity

$$\mathcal{F}_N^2 = \frac{1}{N}\mathcal{R}_N$$

where

$$\mathcal{R}_N := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & & \vdots & \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

is the reflection operator, i.e.,

$$(\mathcal{R}_N f)[n] := f[-n], \qquad n = 0, 1, \dots, N-1,$$

(with arguments taken modulo N). We then find

$$\mathcal{F}_N^4 = \left[\frac{1}{N}\mathcal{R}_N\right]^2 = \frac{1}{N^2}\mathcal{R}_N^2 = \frac{1}{N^2}I_N$$

where I_N is the $N \times N$ identity matrix. In this way we see that if

$$\mathcal{F}_N f = \lambda f, \qquad f \neq 0,$$

then

$$\lambda^4 - \frac{1}{N^2} = 0,$$

so λ must take one of the values $\pm 1/\sqrt{N}, \pm i/\sqrt{N}$.

The columns (and rows) of $\mathcal{F}_N$ are pairwise orhogonal with

$$\overline{\mathcal{F}_N}^T \mathcal{F}_N = \frac{1}{N} I_N = \mathcal{F}_N \overline{\mathcal{F}_N}^T.$$

Thus $\mathcal{F}_N$ is normal so there is an orthonormal basis for $\mathbb{C}^N$ consisting of eigenvectors of $\mathcal{F}_N$, [12, p. 226].

Now if

$$\mathcal{F}_N f = \pm\frac{1}{\sqrt{N}} f, \qquad f \neq 0,$$

then

$$\mathcal{R}_N f = (N\mathcal{F}_N^2) f = N\left(\pm\frac{1}{\sqrt{N}}\right)^2 f = f,$$

i.e., f is even, and if

$$\mathcal{F}_N f = \pm\frac{i}{\sqrt{N}} f, \qquad f \neq 0,$$

then

$$\mathcal{R}_N f = (N\mathcal{F}_N^2) f = N\left(\pm\frac{i}{\sqrt{N}}\right)^2 f = -f,$$

i.e., f is odd. By using Euler's identity we can write

$$\mathcal{F}_N = \mathcal{C}_N - i\mathcal{S}_N$$

where

$$\mathcal{C}_N := \frac{1}{N}\left[\cos\left(2\pi\frac{nk}{N}\right)\right]_{n,k=0}^{N-1} \tag{15}$$

$$\mathcal{S}_N := \frac{1}{N}\left[\sin\left(2\pi\frac{nk}{N}\right)\right]_{n,k=0}^{N-1} \tag{16}$$

are symmetric real matrices. We easily verify that

$$\begin{aligned}\mathcal{C}_N\mathcal{R}_N = \mathcal{C}_N = \mathcal{R}_N\mathcal{C}_N\\ \mathcal{S}_N\mathcal{R}_N = -\mathcal{S}_N = \mathcal{R}_N\mathcal{S}_N,\end{aligned}$$

and thereby conclude that if

$$\mathcal{F}_N f = \pm\frac{1}{\sqrt{N}}f$$

then

$$\mathcal{C}_N f = \pm\frac{1}{\sqrt{N}}f,$$

and if

$$\mathcal{F}_N f = \pm\frac{i}{\sqrt{N}}f$$

then

$$\mathcal{S}_N f = \mp\frac{1}{\sqrt{N}}f.$$

Since $\mathcal{C}_N, \mathcal{S}_N$ are real and symmetric, this guarantes that we can find a real orthonormal basis for $\mathbb{C}^N$ consisting of eigenvectors of $\mathcal{F}_N$. We produce such a basis by finding the eigenvectors of the real matrices $\mathcal{C}_N, \mathcal{S}_N$ that correspond to the real eigenvalues $\pm\frac{1}{\sqrt{N}}$.

In March 1972, J.H. McClellan and T.W. Parks [7, III, p. 67–68], while students in the electrical engineering department of Rice University, studied the eigenvector decomposition of the discrete Fourier transform operator $\mathcal{F}_N$. They constructed an orthogonal basis for the space $\mathbb{C}^N$ consisting of eigenvectors of $\mathcal{F}_N$. They managed to determine the multiplicities of the four possible eigenvalues, $\pm 1/\sqrt{N}, \pm i/\sqrt{N}$ of $\mathcal{F}_N$. A few years later Auslander and Tolimieri [1, Theorem I.1.2', p. 856] derived an alternative algebraic argument, to find the multiplicities of the eigenvalues of $\mathcal{F}_N$. Kammler [3, Ex. 5.17] simplified their trace argument that uses the known value of the Gauss sum

$$N\ \mathrm{tr}\mathcal{F}_N := \sum_{n=0}^{N-1} e^{2\pi i n^2/N} = \sqrt{N}\begin{cases} 1 & N \equiv 1 \mod 4 \\ 0 & N \equiv 2 \mod 4 \\ i & N \equiv 3 \mod 4 \\ 1+i & N \equiv 0 \mod 4, \end{cases}$$

to find a simple formula for the sum of the eigenvalues of $\mathcal{F}_N$, see [3, Chapter 4, p. 215].

Let $M_r(N)$ be the multiplicity of the eigenvalue

$$\lambda = \frac{(-i)^r}{\sqrt{N}}$$

of $\mathcal{F}_N$, $\quad r = 0, 1, 2, 3$. From the above references, we obtain the formulas

$$M_0(N) := \lfloor N/4 \rfloor + 1 \tag{17}$$

$$M_1(N) := \begin{cases} \lfloor N/4 \rfloor & \text{if } N \equiv 0, 1, 2 \mod 4 \\ \lfloor N/4 \rfloor + 1 & \text{if } N \equiv 3 \mod 4, \end{cases} \tag{18}$$

$$M_2(N) := \begin{cases} \lfloor N/4 \rfloor & \text{if } N \equiv 0, 1 \mod 4 \\ \lfloor N/4 \rfloor + 1 & \text{if } N \equiv 2, 3 \mod 4, \end{cases} \tag{19}$$

$$M_3(N) := \begin{cases} \lfloor N/4 \rfloor - 1 & \text{if } N \equiv 0 \mod 4 \\ \lfloor N/4 \rfloor & \text{if } N \equiv 1, 2, 3 \mod 4 \end{cases} \tag{20}$$

for the multiplicities of the eigenvalues $1/\sqrt{N}, -i/\sqrt{N}, -1/\sqrt{N}$, and $i/\sqrt{N}$, respectively. Here $\lfloor\ \rfloor$ is the floor function, i.e., $\lfloor x \rfloor$ is the greatest integer that does not exceed x. Table 1 shows the values of these functions for selected values of N. Since the matrix $\mathcal{F}_N$ is normal, these multiplicities are also the dimensions of the eigenspaces of the operator $\mathcal{F}_N$. We will now construct a real, orthonormal basis for $\mathbb{C}^N$ of eigenvectors of $\mathcal{F}_N$ when $N = 1, 2, 3, 4, 5, 6$. We will let

$$f_{N,r,\mu}[n],\ \mu = 1, 2, \ldots, M_r(N) \tag{21}$$

be orthonormal eigenvectors of $\mathcal{F}_N$ corresponding to the eigenvalue

$$\lambda = \frac{(-i)^r}{\sqrt{N}},\ r = 0, 1, 2, 3.$$

Example 2.1. $N = 1$

The matrix

$$\mathcal{F}_1 = [1]$$

$N\backslash\lambda$	$1/\sqrt{N}$	$-i/\sqrt{N}$	$-1/\sqrt{N}$	$i/\sqrt{N}$
1	1	0	0	0
2	1	0	1	0
3	1	1	1	0
4	2	1	1	0
5	2	1	1	1
6	2	1	2	1
7	2	2	2	1
8	3	2	2	1
4m	m+1	m	m	m-1
4m+1	m+1	m	m	m
4m+2	m+1	m	m+1	m
4m+3	m+1	m+1	m+1	m

TABLE 1. Dimensions of the eigenspaces of the operator $\mathcal{F}_N$.

has the eigenvalue $\lambda_1 = 1$ corresponding to the eigenvector [1]. We set

$$f_{1,0,1}[n] = 1, \qquad n \in \mathbb{Z}.$$

Example 2.2. $N = 2$

The matrix

$$\mathcal{F}_2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 1/\sqrt{2}$, $\lambda_2 = -1/\sqrt{2}$ with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}.$$

We normalize these vectors to obtain

$$f_{2,0,1}[0] = \frac{1}{\sqrt{4-2\sqrt{2}}}, \qquad f_{2,0,1}[1] = \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}},$$
$$f_{2,2,1}[0] = \frac{1}{\sqrt{4+2\sqrt{2}}}, \qquad f_{2,2,1}[1] = -\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}},$$

and then 2-periodicity extend these functions to $\mathbb{Z}$.

N	$f_{N,r,\mu}$	$\sqrt{N}\lambda$	
1	$f_{1,0,1}$	1	(1)
2	$f_{2,0,1}$	1	(0.9239,0.3827)
	$f_{2,2,1}$	-1	(0.3827,-0.9239)
3	$f_{3,0,1}$	1	(0.8881,0.3251,0.3251)
	$f_{3,1,1}$	-i	(0,-0.7071,0.7071)
	$f_{3,2,1}$	-1	(-0.4597,0.6280,0.6280)
4	$f_{4,0,1}$	1	(0.7071,0,0.7071,0)
	$f_{4,0,2}$	1	(0.5,0.5,-0.5,0.5)
	$f_{4,1,1}$	-i	(0,-0.7071,0,0.7071)
	$f_{4,2,1}$	-1	(-0.5,0.5,0.5,0.5)
5	$f_{5,0,1}$	1	(0.6793,-0.0910,0.5109,0.5109,-0.0910)
	$f_{5,0,2}$	1	(0.5120,0.5575,-0.2411,-0.2411,0.5575)
	$f_{5,1,1}$	-i	(0,-0.6802,-0.1932,0.1932,0.6802)
	$f_{5,2,1}$	-1	(-0.5257,0.4253,0.4253,0.4253,0.4253)
	$f_{5,3,1}$	i	(0,-0.1932,0.6802,-0.6802,0.1932)
6	$f_{6,0,1}$	1	(0.7720,0.0116,0.3406,0.4145,0.3406,0.0116)
	$f_{6,0,2}$	1	(-0.3289,-0.5932,0.1787,0.3522,0.1787,-0.5932)
	$f_{6,1,1}$	-i	(0,0.6533,0.2706,0,-0.2706,-0.6533)
	$f_{6,2,1}$	-1	(-0.3830,0.3241,-0.0621,0.7972,-0.0621,0.3241)
	$f_{6,2,2}$	-1	(-0.3862,0.2071,0.5901,-0.2620,0.5901,0.2071)
	$f_{6,3,1}$	i	(0,0.2706,-0.6533,0,0.6533,-0.2706)

TABLE 2. Approximate numerical values for orthonormal eigenvectors of $\mathcal{F}_N$ when $N \leq 6$.

We can numerically find such eigenvectors when $N = 3, 4, 5, 6, \ldots$. Indeed, we create the matrices $\mathcal{C}_N, \mathcal{S}_N$ from (15),(16) and use MATLAB to find orthonormal eigenvectors corresponding to the eigenvalues $\pm\frac{1}{\sqrt{N}}$. In this way we obtain orthonormal eigenvectors $f_{3,r,\mu}$ for $\mathcal{F}_3$, $f_{4,r,\mu}$ for $\mathcal{F}_4$, $f_{5,r,\mu}$ for $\mathcal{F}_5$, and $f_{6,r,\mu}$ for $\mathcal{F}_6$ as shown in Table 2.

3. The main mathematical results

3.1. **Periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}$.** A generalized function f, $f \neq 0$, on $\mathbb{R}$ is said to be an eigenfunction of the Fourier transform operator $\mathcal{F}$ if

$$\mathcal{F}f = \lambda f$$

for $\lambda = \pm 1, \pm i$. We will now determine all periodic eigenfunctions of $\mathcal{F}$.

3.1.1. *Periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}$.* Let f be a p-periodic generalized function on $\mathbb{R}$, $p > 0$, and assume that

$$F := \mathcal{F}f = \lambda f$$

where $\lambda = \pm 1, \pm i$ and $f \neq 0$. Fig. 1 illustrates such a 2-periodic eigenfunction of $\mathcal{F}$,

$$f(x) = \frac{1}{2}\left\{ Ш\left(\frac{x}{2}\right) + Ш\left(\frac{x-1/2}{2}\right) - Ш\left(\frac{x-2/2}{2}\right) + Ш\left(\frac{x-3/2}{2}\right)\right\},$$

constructed from the eigenvector $f_{4,0,2}$ of $\mathcal{F}_4$. We will now characterize all such periodic eigenfunctions.

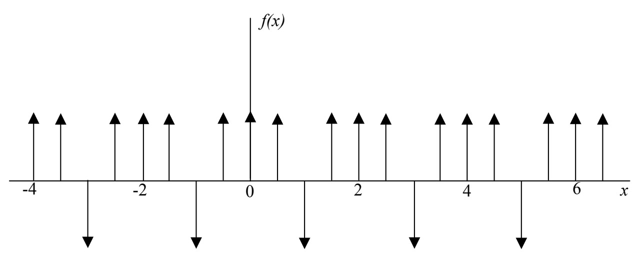

Figure 1. The 2-periodic generalized eigenfunction of the Fourier transform operator $\mathcal{F}$ constructed from $f_{4,0,2}$.

Since f is p-periodic, f is represented by its weakly convergent Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} \Gamma[k] e^{2\pi i k x/p} \tag{22}$$

(see (9) and [3, Chapter 7, p. 440]) with Fourier coefficients $\Gamma[k]$ that are slowly growing as $k \to \pm\infty$, i.e., the sequence

$$\frac{|\Gamma[k]|}{(1+k^2)^m}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

is bounded for some choice of $m = 1, 2, \ldots$. We Fourier transform term by term to obtain the weakly convergent series

$$F(s) = \sum_{k=-\infty}^{\infty} \Gamma[k]\delta\left(s - \frac{k}{p}\right) \tag{23}$$

for the Fourier transform of f. Now since $F = \lambda f$ and $\lambda \neq 0$, F must also be p- periodic with

$$\begin{aligned} F(s) &= \left\{ \sum_{0 \le k < p^2} \Gamma[k]\delta\left(s - \frac{k}{p}\right)\right\} * \sum_{m=-\infty}^{\infty} \delta(s - mp) \\ &= \left\{ \sum_{0 \le k < p^2} \Gamma[k]\delta\left(s - \frac{k}{p}\right)\right\} * \frac{1}{p}\text{Ш}\left(\frac{s}{p}\right). \end{aligned}$$

We recognize this as the Fourier transform of

$$\begin{aligned} f(x) &= \left\{ \sum_{0 \le k < p^2} \Gamma[k] e^{2\pi i k x/p} \right\} \text{Ш}(px) \\ &= \left\{ \sum_{0 \le k < p^2} \Gamma[k] e^{2\pi i k x/p} \right\} \frac{1}{p} \sum_{n=-\infty}^{\infty} \delta\left(x - \frac{n}{p}\right) \\ &= \frac{1}{p} \sum_{n=-\infty}^{\infty} \sum_{0 \le k < p^2} \Gamma[k] e^{2\pi i k x/p} \delta\left(x - \frac{n}{p}\right) \\ &= \frac{1}{p} \sum_{n=-\infty}^{\infty} \left\{ \sum_{0 \le k < p^2} \Gamma[k] e^{2\pi i k n/p^2} \right\} \delta\left(x - \frac{n}{p}\right). \end{aligned}$$

We define

$$\gamma[n] := \sum_{0 \le k < p^2} \Gamma[k] e^{2\pi i k n/p^2}$$

and write

$$f(x) = \frac{1}{p} \sum_{n=-\infty}^{\infty} \gamma[n]\delta\left(x - \frac{n}{p}\right). \tag{24}$$

Now if the term

$$\gamma[n]\delta\left(x - \frac{n}{p}\right), \qquad \gamma[n] \neq 0$$

appears in the sum (24) then (since f is p-periodic)

$$\gamma[n]\delta\left(x - p - \frac{n}{p}\right)$$

must also appear. Thus

$$\gamma[n']\delta\left(x - \frac{n'}{p}\right) = \gamma[n]\delta\left(x - \frac{p^2+n}{p}\right)$$

for some integer n'. It follows that

$$\frac{n'}{p} = \frac{p^2+n}{p},$$

i.e.,

$$p^2 = n' - n,$$

and

$$\gamma[n] = \gamma[n'].$$

Thus

$$p^2 = N$$

for some $N = 1, 2, \ldots$, and since $\gamma[n]$ is N-periodic, we can use (24) to write

$$f(x) = \frac{1}{\sqrt{N}} \sum_{n=-\infty}^{\infty} \gamma[n]\delta\left(x - \frac{n}{\sqrt{N}}\right)$$

$$= \frac{1}{\sqrt{N}} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} \gamma[n]\delta\left(x - \frac{n}{\sqrt{N}} - m\sqrt{N}\right) \tag{25}$$

where

$$\gamma[n] = \sum_{k=0}^{N-1} \Gamma[k] e^{2\pi ikn/N}$$

is the inverse Fourier transform of the N-periodic sequence of Fourier coefficients Γ. Since $F = \lambda f$ we can use (23), (25) to see that

$$\Gamma[k] = (\mathcal{F}_N\ \gamma)[k] = \frac{\lambda}{\sqrt{N}}\gamma[k], \qquad k = 0, 1, \ldots, N-1,$$

i.e., that γ is an eigenvector of the discrete Fourier transform operator $\mathcal{F}_N$ associated with the eigenvalue $\dfrac{\lambda}{\sqrt{N}}$. In this way we prove the following

Theorem 3.1. *Let the generalized function f on $\mathbb{R}$ be a p-periodic eigenfunction of the Fourier transform operator $\mathcal{F}$ with eigenvalue $\lambda = 1, -i, -1,$ or $+i$. Then $p = \sqrt{N}$ for some integer $N = 1, 2, \ldots$ and f has the representation*

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} \gamma[n]\delta\left(x - \frac{n}{p} - mp\right) \tag{26}$$

where γ is an eigenvector of the discrete Fourier transform operator $\mathcal{F}_N$ with

$$(\mathcal{F}_N\ \gamma)[k] = \frac{1}{N}\sum_{n=0}^{N-1}\gamma[n]e^{-2\pi ikn/N} = \frac{\lambda}{\sqrt{N}}\gamma[k], \qquad k = 0, 1, \ldots, N-1.$$

Example 3.1. When $N = 1$ we obtain the corresponding 1-periodic

$$f(x) = \sum_{n=-\infty}^{\infty}\delta(x-n) = Ш(x),$$

with

$$\mathcal{F}Ш = Ш.$$

Of course, this particular result is well known, see [2, p. 152–153]. Our argument shows that a periodic eigenfunction of the Fourier transform operator that has one singular point per unit cell must be a scalar multiple of the Dirac comb Ш. A graphical representation of this f is given in Fig. 2.

Example 3.2. When $N = 2$, we obtain the $\sqrt{2}$-periodic eigenfunctions

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{4-2\sqrt{2}}}\sum_{n=-\infty}^{\infty}\delta(x-n\sqrt{2})\frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{1}{\sqrt{2}}-n\sqrt{2}\Big)\\ &= \frac{1}{\sqrt{2(4-2\sqrt{2})}}\,Ш\Big(\frac{x}{\sqrt{2}}\Big)+\frac{-1+\sqrt{2}}{\sqrt{2(4-2\sqrt{2})}}\,Ш\Big(\frac{x}{\sqrt{2}}-\frac{1}{2}\Big), \end{aligned}$$

and

$$\begin{aligned} f_2(x) &= \frac{1}{\sqrt{4+2\sqrt{2}}}\sum_{n=-\infty}^{\infty}\delta(x-n\sqrt{2})-\frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{1}{\sqrt{2}}-n\sqrt{2}\Big)\\ &= \frac{1}{\sqrt{2(4+2\sqrt{2})}}\,Ш\Big(\frac{x}{\sqrt{2}}\Big)-\frac{1+\sqrt{2}}{\sqrt{2(4+2\sqrt{2})}}\,Ш\Big(\frac{x}{\sqrt{2}}-\frac{1}{2}\Big) \end{aligned}$$

from the eigenvectors $f_{2,0,1}$ and $f_{2,2,1}$ for $\mathcal{F}_2$ as given in Example 3.1.2. It is easy to verify that

$$(\mathcal{F}f_1)(s) = f_1(s), \qquad (\mathcal{F}f_2)(s) = -f_2(s).$$

Graphical representations of f_1, f_2 are shown in Fig. 2. Filled circles correspond to negatively scaled Dirac δ's. The radius of each circle is proportional to the square root of the modulus of the scale factor for the corresponding δ. (When we illumine a circular aperture with light having a uniform intensity, the energy passing through the aperture is proportional to the area of the aperture, i.e., to the square of the radius. The scaling we have chosen for our graphs makes the "intensity" of the δ correspond to the "intensity" associated with the corresponding circular aperture.)

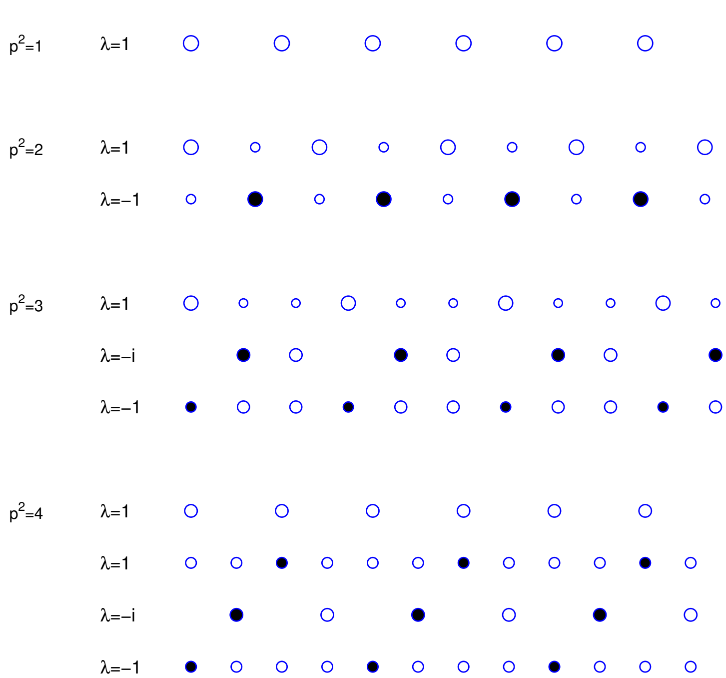

FIGURE 2. Graphical representation of the p-periodic eigenfunctions, of the univariate Fourier transform operator with $p^2 = 1, 2, 3, 4$.

Example 3.3. When $N = 3$ we obtain the $\sqrt{3}$-periodic eigenfunctions

$$\begin{aligned} f_1(x) &= \frac{1+\sqrt{3}}{\sqrt{6+2\sqrt{3}}} \sum_{n=-\infty}^{\infty} \delta(x - n\sqrt{3}) + \frac{1}{\sqrt{6+2\sqrt{3}}} \sum_{n=-\infty}^{\infty} \delta\Big(x - \frac{1}{\sqrt{3}} - n\sqrt{3}\Big) \\ &\quad + \frac{1}{\sqrt{6+2\sqrt{3}}} \sum_{n=-\infty}^{\infty} \delta\Big(x - \frac{2}{\sqrt{3}} - n\sqrt{3}\Big) \\ &= \frac{1+\sqrt{3}}{\sqrt{3(6+2\sqrt{3})}} \,\text{III}\Big(\frac{x}{\sqrt{3}}\Big) + \frac{1}{\sqrt{3(6+2\sqrt{3})}} \,\text{III}\Big(\frac{x}{\sqrt{3}} - \frac{1}{3}\Big) \\ &\quad + \frac{1}{\sqrt{3(6+2\sqrt{3})}} \,\text{III}\Big(\frac{x}{\sqrt{3}} - \frac{2}{3}\Big), \\ f_2(x) &= -\frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \delta\Big(x - \frac{1}{\sqrt{3}} - n\sqrt{3}\Big) + \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} \delta\Big(x - \frac{2}{\sqrt{3}} - n\sqrt{3}\Big) \\ &= -\frac{1}{\sqrt{6}} \,\text{III}\Big(\frac{x}{\sqrt{3}} - \frac{1}{3}\Big) + \frac{1}{\sqrt{6}} \,\text{III}\Big(\frac{x}{\sqrt{3}} - \frac{2}{3}\Big), \end{aligned}$$

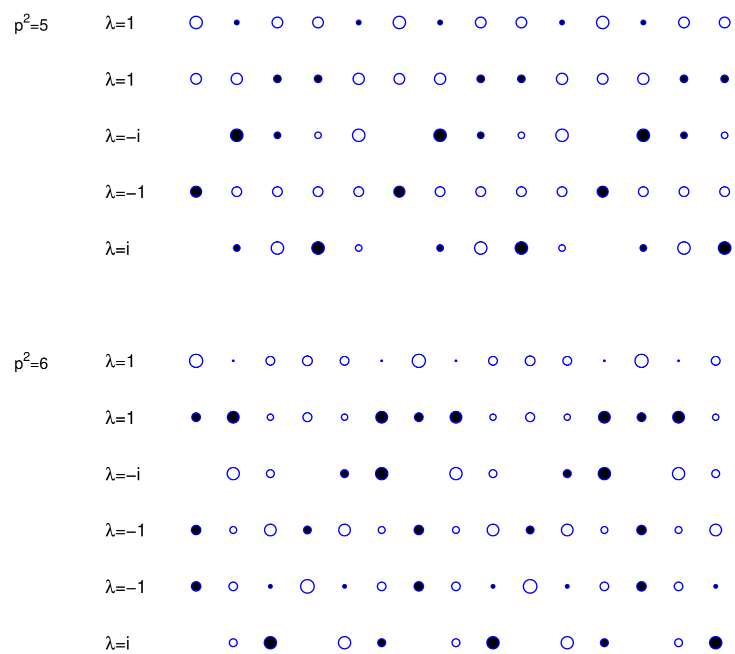

FIGURE 3. Graphical representation of the p-periodic eigenfunctions, of the univariate Fourier transform operator with $p^2 = 5, 6$.

and

$$\begin{aligned} f_3(x) &= \frac{1-\sqrt{3}}{\sqrt{6-2\sqrt{3}}} \sum_{n=-\infty}^{\infty} \delta(x - n\sqrt{3}) + \frac{1}{\sqrt{6-2\sqrt{3}}} \sum_{n=-\infty}^{\infty} \delta\Big(x - \frac{1}{\sqrt{3}} - n\sqrt{3}\Big) \\ &\quad + \frac{1}{\sqrt{6-2\sqrt{3}}} \sum_{n=-\infty}^{\infty} \delta\Big(x - \frac{2}{\sqrt{3}} - n\sqrt{3}\Big) \\ &= \frac{1-\sqrt{3}}{\sqrt{3(6-2\sqrt{3})}} \,\text{III}\Big(\frac{x}{\sqrt{3}}\Big) + \frac{1}{\sqrt{3(6-2\sqrt{3})}} \,\text{III}\Big(\frac{x}{\sqrt{3}} - \frac{1}{3}\Big) \\ &\quad + \frac{1}{\sqrt{3(6-2\sqrt{3})}} \,\text{III}\Big(\frac{x}{\sqrt{3}} - \frac{2}{3}\Big) \end{aligned}$$

from the eigenvectors $f_{3,0,1}$,$f_{3,1,1}$ and $f_{3,2,1}$ for $\mathcal{F}_3$. Graphical representations of f_1, f_2, f_3 are shown in Fig. 2. With a bit of calculation, we can verify that

$$(\mathcal{F} f_1)(s) = f_1(s), \qquad (\mathcal{F} f_2)(s) = -i\ f_2(s), \qquad (\mathcal{F} f_3)(s) = -f_3(s).$$

(The eigenvalue $+i$ does not appear when $N=3$ and $p=\sqrt{3}$.)

Example 3.4. When $N=4$ we obtain the 2-periodic eigenfunctions

$$\begin{aligned} f_1(x) &= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\delta(x-2n)+\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\delta(x-1-2n)\\ &= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\delta(x-n)\\ &= \frac{1}{\sqrt{2}}\,\text{Ш}(x), \end{aligned}$$

$$\begin{aligned} f_2(x) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta(x-2n)+\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{1}{2}-2n\Big)\\ &\quad -\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta(x-1-2n)+\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{3}{2}-2n\Big)\\ &= \frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}\Big)+\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}-\frac{1}{4}\Big)-\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}-\frac{2}{4}\Big)+\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}-\frac{3}{4}\Big), \end{aligned}$$

$$\begin{aligned} f_3(x) &= -\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{1}{2}-2n\Big)+\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{3}{2}-2n\Big)\\ &= -\frac{1}{2\sqrt{2}}\,\text{Ш}\Big(\frac{x}{2}-\frac{1}{4}\Big)+\frac{1}{2\sqrt{2}}\,\text{Ш}\Big(\frac{x}{2}-\frac{3}{4}\Big), \end{aligned}$$

and

$$\begin{aligned} f_4(x) &= -\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta(x-2n)+\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{1}{2}-2n\Big)\\ &\quad +\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta(x-1-2n)+\frac{1}{2}\sum_{n=-\infty}^{\infty}\delta\Big(x-\frac{3}{2}-2n\Big)\\ &= -\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}\Big)+\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}-\frac{1}{4}\Big)+\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}-\frac{2}{4}\Big)+\frac{1}{4}\,\text{Ш}\Big(\frac{x}{2}-\frac{3}{4}\Big) \end{aligned}$$

from the eigenvectors $f_{4,0,1}$,$f_{4,0,2}$,$f_{4,1,1}$ and $f_{4,2,1}$ for $\mathcal{F}_4$. Graphical representations of f_1, f_2, f_3, f_4 are shown in Fig. 2. We can easily verify that

$$\begin{aligned} (\mathcal{F}f_1)(s) &= f_1(s), & (\mathcal{F}f_2)(s) &= f_2(s),\\ (\mathcal{F}f_3)(s) &= -i\ f_3(s), & (\mathcal{F}f_4)(s) &= -\ f_4(s). \end{aligned}$$

(The eigenvalue $+i$ does not appear when $N=4$ and $p=2$, see Table 1. The eigenfunction f_1 is 1-periodic as well as 2-periodic.) Fig. 3 shows representations for $\sqrt{5}$-periodic and $\sqrt{6}$-periodic eigenfunctions of $\mathcal{F}$ constructed from the 5 eigenvectors of $\mathcal{F}_5$ and from the 6 eigenvectors of $\mathcal{F}_6$ (as given in Table 2), respectively.

3.2. **Periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^2$.**

3.2.1. *Eigenfunctions of* $\mathcal{F}_{N_1,N_2}$. We will generalize the results of Section 3.1 to 2 dimension. For this purpose, will need all doubly periodic eigenfunctions of the bivariate discrete Fourier transform operator.

Let N_1, N_2 be positive integers and let f be a complex-valued function on $\mathbb{Z}\times\mathbb{Z}$ that is N_1-periodic in the first component and N_2-periodic in the second, i.e.,

$$\begin{aligned} f[n_1+N_1,n_2] &= f[n_1,n_2] \qquad \text{for all } n_1,n_2\in\mathbb{Z},\\ f[n_1,n_2+N_2] &= f[n_1,n_2] \qquad \text{for all } n_1,n_2\in\mathbb{Z}. \end{aligned}$$

We will say that f is a function on $\mathbb{P}_{N_1,N_2}$. The collection of all such functions is a complex linear space of dimension N_1N_2.

Definition 3.2. The discrete Fourier transform operator $\mathcal{F}_{N_1,N_2}$ on $\mathbb{P}_{N_1,N_2}$ is defined by

$$(\mathcal{F}_{N_1,N_2}f)[k_1,k_2] := \frac{1}{N_1N_2}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} f[n_1,n_2]e^{-2\pi i(k_1n_1/N_1+k_2n_2/N_2)},$$

for $k_1,k_2\in\mathbb{Z}$.

From the definition, we immediatly obtain the tensor product factorizations

$$\mathcal{F}_{N_1,N_2} = \mathcal{F}_{N_1,*}\ \mathcal{F}_{*,N_2} = \mathcal{F}_{*,N_2}\ \mathcal{F}_{N_1,*}$$

and

$$\mathcal{F}^{\,p}_{N_1,N_2} = \mathcal{F}^{\,p}_{N_1,*}\ \mathcal{F}^{\,p}_{*,N_2}, \qquad p=1,2,\dots$$

where

$$\begin{aligned} (\mathcal{F}_{N_1,*}f)[k_1,n_2] &= \frac{1}{N_1}\sum_{n_1=0}^{N_1-1} f[n_1,n_2]e^{-2\pi i n_1k_1/N_1}\\ (\mathcal{F}_{*,N_2}f)[n_1,k_2] &= \frac{1}{N_2}\sum_{n_2=0}^{N_2-1} f[n_1,n_2]e^{-2\pi i n_2k_2/N_2} \end{aligned}$$

are the corresponding univariate operators that act on the first, second coordinate, respectively.

From our discussion of the univariate operator $\mathcal{F}_N$, we know that

$$\mathcal{F}^4_{N_1,*} = \frac{1}{N_1^2}I_{N_1}, \qquad \mathcal{F}^4_{*,N_2} = \frac{1}{N_2^2}I_{N_2}$$

so we must have

$$\mathcal{F}^4_{N_1,N_2} = \frac{1}{N_1^2N_2^2}I_{N_1,N_2}$$

where $I_{N_1}, I_{N_2}, I_{N_1,N_2}$ are the identity operators for functions on $\mathbb{P}_{N_1}$, $\mathbb{P}_{N_2}, \mathbb{P}_{N_1,N_2}$, respectively. In this way we see that the eigenvalues of $\mathcal{F}_{N_1,N_2}$ are restricted to the values

$$\lambda = \pm 1/\sqrt{N_1 N_2}, \qquad \pm i/\sqrt{N_1 N_2}.$$

Suppose now that f_1, f_2 are eigenfunctions of $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}$, respectively, with corresponding eigenvalues λ_1, λ_2. If we set

$$f[n_1, n_2] := f_1[n_1]\ f_2[n_2]$$

then we can write

$$\begin{aligned}(\mathcal{F}_{N_1,N_2} f)[k_1, k_2] &= (\mathcal{F}_{N_1,*}\mathcal{F}_{*,N_2}) f_1[n_1] f_2[n_2] \\ &= (\mathcal{F}_{N_1,*} f_1)[k_1](\mathcal{F}_{*,N_2} f_2)[k_2] \\ &= \lambda_1 f_1[k_1]\ \lambda_2 f_2[k_2] \\ &= \lambda_1 \lambda_2 f[k_1, k_2].\end{aligned}$$

In this way we see that f is an eigenfunction of $\mathcal{F}_{N_1,N_2}$ associated with the eigenvalue $\lambda_1 \lambda_2$.

More generally, let

$$f_{1,1}[n_1], f_{1,2}[n_1], \ldots, f_{1,N_1}[n_1]$$

be an orthonormal set of eigenfunctions of $\mathcal{F}_{N_1}$ with corresponding eigenvalues

$$\lambda_{1,1}, \lambda_{1,2}, \ldots, \lambda_{1,N_1},$$

respectively, and let

$$f_{2,1}[n_2], f_{2,2}[n_2], \ldots, f_{2,N_2}[n_2]$$

be an orthonormal set of eigenfunctions of $\mathcal{F}_{N_2}$ with corresponding eigenvalues

$$\lambda_{2,1}, \lambda_{2,2}, \ldots, \lambda_{2,N_2},$$

respectively. The $N_1 N_2$ functions

$$f_{r_1,r_2}[n_1, n_2] := f_{1,r_1}[n_1] f_{2,r_2}[n_2], \quad r_k = 1, 2, \ldots, N_k, k = 1, 2$$

will be an orthonormal basis for $\mathbb{P}_{N_1,N_2}$ with

$$\mathcal{F}_{N_1,N_2} f_{r_1,r_2}[k_1, k_2] = \lambda_{1,r_1} \lambda_{2,r_2}\ f_{r_1,r_2}[k_1, k_2].$$

Let $M_r(N_1, N_2)$ be the multiplicity of the eigenvalue

$$\lambda = \frac{(-i)^r}{\sqrt{N_1 N_2}}$$

for $\mathcal{F}_{N_1,N_2}$, $r = 0, 1, 2, 3$. Since we know the dimensions of the eigenspaces for $\mathcal{F}_{N_1}, \mathcal{F}_{N_2}$ as given in (17)-(20), we can determine the dimensions of the four eigenspaces of $\mathcal{F}_{N_1,N_2}$ by writing

$$M_r(N_1, N_2) := \sum_{r_1+r_2 \equiv r \pmod 4} M_{r_1}(N_1) M_{r_2}(N_2), \quad r = 0, 1, 2, 3.$$

Table 3, and Table 4 show simplified expressions for these multiplicities. Explicit values for the cases where $N_1 \leq 4$, $N_2 \leq 4$, are given in Table 5.

The $N_1 N_2$ eigenfunctions

$$f_{N_1,r_1,\mu_1;N_2,r_2,\mu_2}[n_1, n_2] := f_{N_1,r_1,\mu_1}[n_1] \; f_{N_2,r_2,\mu_2}[n_2], \tag{27}$$

with $\mu_k = 1, \ldots, M_{r_k}(N_k), k = 1, 2$ of $\mathcal{F}_{N_1,N_2}$ serve as an orthonormal basis for the $N_1 N_2$ dimensional space $\mathbb{P}_{N_1,N_2}$. Here (27) has the corresponding eigenvalue

$$\lambda = \frac{(-i)^{r_1}}{\sqrt{N_1}} \frac{(-i)^{r_2}}{\sqrt{N_2}}, \qquad r_1, r_2 = 0, 1, 2, 3.$$

3.2.2. *Periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^2$.*

Definition 3.3. Let f be a function on $\mathbb{R}^2$. The Fourier transform operator $\mathcal{F}$ is defined so that $\mathcal{F}f$ is the functional

$$(\mathcal{F}f)\{\phi\} := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s_1, s_2) \phi^\wedge(s_1, s_2) ds_1 ds_2, \qquad \phi \in \mathcal{S}(\mathbb{R}^2)$$

where

$$\phi^\wedge(s_1, s_2) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i(s_1 x_1 + s_2 x_2)} \phi(x_1, x_2) dx_1 dx_2$$

is the ordinary Fourier transform of ϕ, and

$$\mathcal{S}(\mathbb{R}^2) := \{\phi \in C^\infty(\mathbb{R}^2) : \sup \|x\|^m \|D_1^{n_1} D_2^{n_2} \phi(x)\| < \infty, \; m, n_1, n_2 = 0, 1, 2, \ldots\},$$

where D_1, D_2 represent partial derivatives with respect to x_1, x_2, respectively.

In this bivariate case we again have

$$\phi^{\wedge\wedge\wedge\wedge} = \phi$$

i.e., we have

$$\mathcal{F}^4 = \mathcal{I}$$

so the eigenvalues of $\mathcal{F}$ are again restricted to the values $\lambda = 1, -i, -1$, or $+i$. We will characterize the eigenfunctions of the operator $\mathcal{F}$ that are doubly periodic on $\mathbb{R}^2$. We will adopt vector notation for the remaining of the section.

$$\lambda = \frac{1}{\sqrt{N_1 N_2}}$$

$N_1\backslash N_2$	$4m_2$	$4m_2+1$
$4m_1$	$4m_1m_2+1$	$4m_1m_2+m_1+1$
$4m_1+1$	$4m_1m_2+m_2+1$	$4m_1m_2+m_1+m_2+1$
$4m_1+2$	$4m_1m_2+2m_2+1$	$4m_1m_2+2m_2+m_1+1$
$4m_1+3$	$4m_1m_2+3m_2$	$4m_1m_2+3m_2+m_1+1$

$N_1\backslash N_2$	$4m_2+2$	$4m_2+3$
$4m_1$	$4m_1m_2+2m_1+1$	$4m_1m_2+3m_1$
$4m_1+1$	$4m_1m_2+2m_1+m_2+1$	$4m_1m_2+3m_1+m_2+1$
$4m_1+2$	$4m_1m_2+2m_1+2m_2+2$	$4m_1m_2+3m_1+2m_2+2$
$4m_1+3$	$4m_1m_2+3m_2+2m_1+2$	$4m_1m_2+3m_1+3m_2+2$

$$\lambda = -\frac{i}{\sqrt{N_1 N_2}}$$

$N_1\backslash N_2$	$4m_2$	$4m_2+1$
$4m_1$	$4m_1m_2$	$4m_1m_2+m_1$
$4m_1+1$	$4m_1m_2+m_2$	$4m_1m_2+m_1+m_2$
$4m_1+2$	$4m_1m_2+2m_2-1$	$4m_1m_2+2m_2+m_1$
$4m_1+3$	$4m_1m_2+3m_2$	$4m_1m_2+3m_2+m_1+1$

$N_1\backslash N_2$	$4m_2+2$	$4m_2+3$
$4m_1$	$4m_1m_2+2m_1-1$	$4m_1m_2+3m_1$
$4m_1+1$	$4m_1m_2+2m_1+m_2$	$4m_1m_2+3m_1+m_2+1$
$4m_1+2$	$4m_1m_2+2m_1+2m_2$	$4m_1m_2+3m_1+2m_2+1$
$4m_1+3$	$4m_1m_2+3m_2+2m_1+1$	$4m_1m_2+3m_1+3m_2+2$

$$\lambda = -\frac{1}{\sqrt{N_1 N_2}}$$

$N_1\backslash N_2$	$4m_2$	$4m_2+1$
$4m_1$	$4m_1m_2+1$	$4m_1m_2+m_1$
$4m_1+1$	$4m_1m_2+m_2$	$4m_1m_2+m_1+m_2$
$4m_1+2$	$4m_1m_2+2m_2+1$	$4m_1m_2+2m_2+m_1+1$
$4m_1+3$	$4m_1m_2+3m_2+1$	$4m_1m_2+3m_2+m_1+1$

$N_1\backslash N_2$	$4m_2+2$	$4m_2+3$
$4m_1$	$4m_1m_2+2m_1+1$	$4m_1m_2+3m_1+1$
$4m_1+1$	$4m_1m_2+2m_1+m_2+1$	$4m_1m_2+3m_1+m_2+1$
$4m_1+2$	$4m_1m_2+2m_1+2m_2+2$	$4m_1m_2+3m_1+2m_2+2$
$4m_1+3$	$4m_1m_2+3m_2+2m_1+2$	$4m_1m_2+3m_1+3m_2+3$

TABLE 3. Dimension of the eigenspace of the operator $\mathcal{F}_{N_1,N_2}$.

3.2.3. *Characterization of periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^2$.* Let f be a bivariate generalized function and assume that f is an eigenfunction of $\mathcal{F}$, i.e.,

$$F := \mathcal{F}f = \lambda f$$

$$\lambda = -\frac{1}{\sqrt{N_1 N_2}}$$

$N_1 \backslash N_2$	$4m_2$	$4m_2+1$
$4m_1$	$4m_1m_2-2$	$4m_1m_2+m_1-1$
$4m_1+1$	$4m_1m_2+m_2-1$	$4m_1m_2+m_1+m_2$
$4m_1+2$	$4m_1m_2+2m_2-1$	$4m_1m_2+2m_2+m_1$
$4m_1+3$	$4m_1m_2+3m_2-1$	$4m_1m_2+3m_2+m_1$

$N_1 \backslash N_2$	$4m_2+2$	$4m_2+3$
$4m_1$	$4m_1m_2+2m_1-1$	$4m_1m_2+3m_1-1$
$4m_1+1$	$4m_1m_2+2m_1+m_2$	$4m_1m_2+3m_1+m_2$
$4m_1+2$	$4m_1m_2+2m_1+2m_2$	$4m_1m_2+3m_1+2m_2+1$
$4m_1+3$	$4m_1m_2+3m_2+2m_1+1$	$4m_1m_2+3m_1+3m_2+2$

TABLE 4. Dimension of the eigenspace of the operator $\mathcal{F}_{N_1,N_2}$ continued.

$$\lambda = \frac{1}{\sqrt{N_1 N_2}}$$

$N_1 \backslash N_2$	1	2	3	4
1	1	1	1	2
2	1	2	2	3
3	1	2	2	3
4	2	3	3	5

$$\lambda = -\frac{1}{\sqrt{N_1 N_2}}$$

$N_1 \backslash N_2$	1	2	3	4
1	0	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	5

$$\lambda = -\frac{i}{\sqrt{N_1 N_2}}$$

$N_1 \backslash N_2$	1	2	3	4
1	0	0	1	1
2	0	0	1	1
3	1	1	2	3
4	1	1	3	4

$$\lambda = \frac{i}{\sqrt{N_1 N_2}}$$

$N_1 \backslash N_2$	1	2	3	4
1	0	0	0	0
2	0	0	1	1
3	0	1	2	2
4	0	1	2	2

TABLE 5. Dimension of the eigenspace of the operator $\mathcal{F}_{N_1,N_2}$ when $N_1, N_2 \leq 4$.

with $\lambda = 1, -i, -1$, or $+i$, (and $f \neq 0$). Assume further that f is a_1, a_2-periodic, i.e.,

$$f(x + a_1) = f(x), \qquad f(x + a_2) = f(x).$$

Here a_1, a_2 are linearly independent vectors in $\mathbb{R}^2$.

We simplify the analysis by rotating the coordinate system as necessary so as to place a shortest vector from the lattice $\mathcal{L}_{a_1,a_2}$ along the positive x-axis. We can and do further assume with no loss of generality that a_1, a_2 have the form

$$a_1 = (\alpha_1, 0)^T, \qquad a_2 = (\beta_1, \beta_2)^T$$

where

$$\alpha_1 > 0 \tag{28}$$

$$\alpha_1^2 \leq \beta_1^2 + \beta_2^2 \tag{29}$$

$$\beta_2 > 0 \tag{30}$$

$$0 \leq \beta_1 < \alpha_1. \tag{31}$$

The dual vectors then have the representation

$$A_1 = \frac{1}{\alpha_1\beta_2}(\beta_2, -\beta_1)^T, \qquad A_2 = \frac{1}{\alpha_1\beta_2}(0, \alpha_1)^T,$$

and

$$\begin{aligned}\text{grid}_{a_1,a_2}(x) &= \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\delta(x - n_1a_1 - n_2a_2)\\ &= \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\delta(x_1 - n_1\alpha_1 - n_2\beta_1, x_2 - n_2\beta_2)\end{aligned}$$

has the Fourier transform

$$\begin{aligned}\text{grid}^{\wedge}_{a_1,a_2}(s) &= |\det{(A_1, A_2)}|\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta(s - k_1A_1 - k_2A_2)\\ &= \frac{1}{\alpha_1\beta_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\delta\left(s_1 - \frac{k_1\beta_2}{\alpha_1\beta_2}, s_2 + \frac{k_1\beta_1 - k_2\alpha_1}{\alpha_1\beta_2}\right).\end{aligned}$$

Now since f is a_1, a_2-periodic, f can be represented by the weakly convergent Fourier series

$$f(x) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\Gamma[k_1,k_2]e^{2\pi i x.(\mathbf{k_1A_1+k_2A_2})}. \tag{32}$$

The Fourier coefficients $\Gamma[k_1,k_2]$, as in the one dimensional case, are slowly growing as $k_1, k_2 \to \pm\infty$, i.e.,

$$\frac{|\Gamma[k_1,k_2]|}{(1+k_1^2+k_2^2)^m}, \qquad k_1, k_2 = 0, \pm 1, \pm 2, \ldots$$

is bounded for some choice of $m = 1, 2, \ldots$. We Fourier transform the series (32) to obtain the weakly convergent series

$$F(s) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\Gamma[k_1,k_2]\delta(s - k_1A_1 - k_2A_2). \tag{33}$$

From (33), we see that the support of F lies on the lattice $\mathcal{L}_{A_1,A_2}$ and since $F = \lambda f$, F must also be a_1, a_2-periodic so we can write

$$F(s) = \left\{ \sum_{k_1A_1+k_2A_2 \in \mathcal{U}} \Gamma[k_1,k_2]\delta(s - k_1A_1 - k_2A_2) \right\} * \mathrm{grid}_{a_1,a_2}(s) \tag{34}$$

where

$$\mathcal{U} := \{x_1^{'}a_1 + x_2^{'}a_2 : 0 \le x_1^{'} < 1, 0 \le x_2^{'} < 1\}$$

is a primitive unit cell associated with the lattice $\mathcal{L}_{a_1,a_2}$, where $x_1^{'}, x_2^{'}$ are affine coordinates, and $*$ is the bivariate convolution product. Using the bivariate inverse Fourier transform, we see that

$$\begin{aligned}
f(x) &= \left\{ \sum_{k_1A_1+k_2A_2 \in \mathcal{U}} \Gamma[k_1,k_2]\; e^{2\pi i(k_1A_1+k_2A_2).\mathbf{x}} \right\} |\det(A_1,A_2)|\; \mathrm{grid}_{A_1,A_2}(x) \\
&= |\det(A_1,A_2)| \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{k_1A_1+k_2A_2 \in \mathcal{U}} \Big\{ \Gamma[k_1,k_2] e^{2\pi i(k_1A_1+k_2A_2).(\mathbf{n_1A_1+n_2A_2})} \\
&\cdot \delta(x - n_1A_1 - n_2A_2) \Big\} \\
&= \frac{1}{\alpha_1\beta_2} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \Bigg\{ \Bigg\{ \sum_{k_1A_1+k_2A_2 \in \mathcal{U}} \Gamma[k_1,k_2] \\
&\cdot e^{2\pi i\{(\beta_1^2+\beta_2^2)n_1k_1 - \alpha_1\beta_1(n_1k_2+n_2k_1) + \alpha_1^2 n_2k_2\}/(\alpha_1^2\beta_2^2)} \Bigg\} \\
&\cdot \delta\left(x_1 - \frac{n_1\beta_2}{\alpha_1\beta_2}, x_2 + \frac{n_1\beta_1 - n_2\alpha_1}{\alpha_1\beta_2}\right) \Bigg\}.
\end{aligned}$$

We define

$$\gamma[n_1,n_2] := \sum_{k_1A_1+k_2A_2 \in \mathcal{U}} \Gamma[k_1,k_2] e^{2\pi i\{(\beta_1^2+\beta_2^2)n_1k_1 - \alpha_1\beta_1(n_1k_2+n_2k_1) + \alpha_1^2 n_2k_2\}/(\alpha_1^2\beta_2^2)} \tag{35}$$

and write

$$f(x_1,x_2) = \frac{1}{\alpha_1\beta_2} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \gamma[n_1,n_2]\delta\left(x_1 - \frac{n_1\beta_2}{\alpha_1\beta_2}, x_2 + \frac{n_1\beta_1 - n_2\alpha_1}{\alpha_1\beta_2}\right). \tag{36}$$

Now f is a_1, a_2-periodic, so if $\gamma[n_1,n_2] \neq 0$ for some integers n_1, n_2, then

$$\begin{aligned}
&\gamma[n_1,n_2]\delta\left(x_1 - \alpha_1 - \frac{n_1\beta_2}{\alpha_1\beta_2}, x_2 + \frac{n_1\beta_1 - n_2\alpha_1}{\alpha_1\beta_2}\right) \\
&\qquad = \gamma[n_1^{'},n_2^{'}]\delta\left(x_1 - \frac{n_1^{'}\beta_2}{\alpha_1\beta_2}, x_2 + \frac{n_1^{'}\beta_1 - n_2^{'}\alpha_1}{\alpha_1\beta_2}\right) \\
&\gamma[n_1,n_2]\delta\left(x_1 - \beta_1 - \frac{n_1\beta_2}{\alpha_1\beta_2}, x_2 - \beta_2 + \frac{n_1\beta_1 - n_2\alpha_1}{\alpha_1\beta_2}\right) \\
&\qquad = \gamma[n_1^{''},n_2^{''}]\delta\left(x_1 - \frac{n_1^{''}\beta_2}{\alpha_1\beta_2}, x_2 + \frac{n_1^{''}\beta_1 - n_2^{''}\alpha_1}{\alpha_1\beta_2}\right)
\end{aligned}$$

for some integers n_1', n_2', n_1'', n_2''. From the supports of these δ-functions we see that

$$\alpha_1 + \frac{n_1\beta_2}{\alpha_1\beta_2} = \frac{n_1'\beta_2}{\alpha_1\beta_2},$$

i.e.,

$$\alpha_1^2 = n_1' - n_1$$
$$\alpha_1^2 = N_1$$

for some $N_1 = 1, 2, \ldots$. Likewise, we see in turn that

$$n_1\beta_1 - n_2\alpha_1 = n_1'\beta_1 - n_2'\alpha_1,$$
$$(n_1' - n_1)\beta_1 = (n_2' - n_2)\alpha_1,$$
$$\alpha_1^2\beta_1 = (n_2' - n_2)\alpha_1,$$
$$\alpha_1\beta_1 = n_2' - n_2 = M$$

for some $M = 0, \pm 1, \pm 2, \ldots$, and analogously

$$\beta_1 + \frac{n_1\beta_2}{\alpha_1\beta_2} = \frac{n_1''\beta_2}{\alpha_1\beta_2},$$
$$\alpha_1\beta_1 = n_1'' - n_1 = M.$$

Finally,

$$\beta_2 - \frac{n_1\beta_1 - n_2\alpha_1}{\alpha_1\beta_2} = -\frac{n_1''\beta_1 - n_2''\alpha_1}{\alpha_1\beta_2},$$
$$\beta_2^2 + (n_1'' - n_1)\frac{\beta_1}{\alpha_1} = n_2'' - n_2,$$
$$\beta_2^2 + \alpha_1\beta_1\,\frac{\beta_1}{\alpha_1} = n_2'' - n_2,$$
$$\beta_2^2 + \beta_1^2 = N_2$$

for some $N_2 = 1, 2, \ldots$. Using these expressions we can now write

$$\alpha_1 = \sqrt{N_1}, \qquad \beta_1 = \frac{M}{\sqrt{N_1}}, \qquad \beta_2 = \frac{\sqrt{N_1N_2 - M^2}}{\sqrt{N_1}}, \quad \text{and}$$
$$a_1 = \frac{1}{\sqrt{N_1}}(N_1, 0)^T, \qquad a_2 = \frac{1}{\sqrt{N_1}}(M, \sqrt{N_1N_2 - M^2})^T,$$
$$A_1 = \frac{1}{\sqrt{N_1(N_1N_2 - M^2)}}(\sqrt{N_1N_2 - M^2}, -M)^T, \; A_2 = \frac{1}{\sqrt{N_1(N_1N_2 - M^2)}}(0, N_1)^T,$$

where, in view of (28)−(31)

$$N_1 \le N_2, \quad 0 \le M < N_1$$

and

$$\|a_1\| = \sqrt{N_1}, \quad \|a_2\| = \sqrt{N_2}.$$

From (35), (33) we also have

$$\gamma[n_1, n_2] = \sum_{k_1 A_1 + k_2 A_2 \in \mathcal{U}} \Gamma[k_1, k_2] e^{2\pi i \{N_2 n_1 k_1 - M(n_1 k_2 + n_2 k_1) + N_1 n_2 k_2 / (N_2 N_1 - M^2)\}} \tag{37}$$

(38)

$$\begin{aligned} F(s) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \Gamma[k_1, k_2] \delta(s - k_1 A_1 - k_2 A_2) \\ &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \Gamma[k_1, k_2] \delta\left(s_1 - \frac{k_1}{\sqrt{N_1}}, s_2 + \frac{k_1 M - k_2 N_1}{\sqrt{N_1 (N_1 N_2 - M^2)}} \right). \end{aligned}$$

We will now consider separately the cases $M = 0, M > 0$.

Case $M = 0$

When $M = 0$ the vectors a_1, a_2 are orthogonal and f has the corresponding periods

$$\alpha_1 = \sqrt{N_1}, \qquad \beta_2 = \sqrt{N_2},$$

along the x-axis and y-axis, respectively. The function γ is represented by the synthesis equation

$$\gamma[n_1, n_2] = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \Gamma[k_1, k_2] e^{2\pi i (n_1 k_1 / N_1 + n_2 k_2 / N_2)}, \tag{39}$$

and by using (38) and (36), in turn we write

$$\begin{aligned} F(s_1, s_2) &= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \Gamma[k_1, k_2] \delta\left(s_1 - \frac{k_1}{\sqrt{N_1}}, s_2 - \frac{k_2}{\sqrt{N_2}} \right), \\ &= \lambda f(s_1, s_2) \\ &= \frac{\lambda}{\sqrt{N_1 N_2}} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \gamma[n_1, n_2] \delta\left(s_1 - \frac{k_1}{\sqrt{N_1}}, s_2 - \frac{k_2}{\sqrt{N_2}} \right). \end{aligned}$$

In this way we conclude that

$$\Gamma[k_1, k_2] = \frac{\lambda}{\sqrt{N_1 N_2}} \gamma[k_1, k_2]. \tag{40}$$

Thus γ must be an eigenvector of the bivariate discrete Fourier transform $\mathcal{F}_{N_1, N_2}$ associated with the eigenvalue $\dfrac{\lambda}{\sqrt{N_1 N_2}}$, $(\lambda = 1, -i, -1$, or $+i)$. Since γ is an N_1, N_2-periodic sequence of complex numbers, we can write

$$f(x) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \Big\{ \gamma[n_1, n_2] \cdot \delta\Big(x_1 - \frac{n_1}{\sqrt{N_1}} - m_1\sqrt{N_1}, x_2 - \frac{n_2}{\sqrt{N_2}} - m_2\sqrt{N_2} \Big) \Big\}.$$

Case $M \neq 0$

We observe that

$$\begin{aligned} a_1 &= \frac{1}{\sqrt{N_1}}(N_1, 0)^T, \\ N_1 a_2 - M a_1 &= \sqrt{N_1}(M, \sqrt{N_1 N_2 - M^2})^T - M(\sqrt{N_1}, 0)^T \\ &= \frac{1}{\sqrt{N_1}}(0, N_1\sqrt{N_1 N_2 - M^2})^T. \end{aligned}$$

Since f is a_1, a_2-periodic, then f is also $a_1, N_1 a_2 - M a_1$-periodic. Thus f has the periods

$$\alpha_1 = \sqrt{N_1}, \quad \text{and} \quad \beta_2' = \sqrt{N_1(N_1 N_2 - M^2)}$$

along the x-axis and the y-axis, respectively, a situation covered by the analysis from the $M = 0$ case. In this way we prove

Theorem 3.2. *Let the generalized function f on $\mathbb{R}^2$ be an a_1, a_2-periodic eigenfunction of the Fourier transform operator $\mathcal{F}$ with eigenvalue $\lambda = 1, -i, -1,$ or $+i$. Assume that the linearly independent periods a_1, a_2 from $\mathbb{R}^2$ have been chosen as small as possible subject to the constraint that $0 < \|a_1\| \leq \|a_2\|$. Then there are positive integers $N_1 \leq N_2$ such that*

$$\|a_1\| = \sqrt{N_1}, \quad \|a_2\| = \sqrt{N_2}$$

and there is a nonnegative integer $M < N_1$ such that a_1 is orthogonal to

$$a_2' := N_1 a_2 - M a_1$$

with

$$\|a_2'\| = \sqrt{N_2'}, \qquad N_2' := N_1(N_1 N_2 - M^2).$$

The generalized function f is a_1, a_2'-periodic and there is an orthogonal transformation Q such that

$$f_Q(x) := f(Qx)$$

is $(\sqrt{N_1}, 0)^T, (0, \sqrt{N_2'})^T$*-periodic with the representation*

$$f_Q(x) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2'-1} \gamma[n_1, n_2]$$

$$\cdot\, \delta\Big(x_1 - \frac{n_1}{\sqrt{N_1}} - m_1\sqrt{N_1}, x_2 - \frac{n_2}{\sqrt{N_2'}} - m_2\sqrt{N_2'}\Big). \tag{41}$$

Here γ *is an eigenfunction of* $\mathcal{F}_{N_1,N_2'}$ *with*

$$(\mathcal{F}_{N_1,N_2'}\ \gamma)[k_1, k_2] = \frac{1}{N_1 N_2'} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2'-1} \gamma[n_1, n_2] e^{-2\pi i(k_1 n_1/N_1 + k_2 n_2/N_2')}$$
$$= \frac{\lambda}{\sqrt{N_1 N_2'}} \gamma[k_1, k_2]$$

for $k_1 = 0, 1, \ldots, N_1 - 1,\ k_2 = 0, 1, \ldots, N_2' - 1.$

Example 3.5. Fig. 4−Fig. 10 display some of the doubly periodic eigenfunctions of the bivariate Fourier transform operator $\mathcal{F}$ that have the form (41) for several values of N_1, N_2 and for $M = 0$. The eigenfunctions γ all have the form

$$\gamma[n_1, n_2] := f_{N_1,r_1,\mu_1;N_2,r_2,\mu_2}[n_1, n_2] = f_{N_1,r_1,\mu_1}[n_1]\ f_{N_2,r_2,\mu_2}[n_2]$$

using $f_{N_1,r_1,\mu_1}[n_1]$ and $f_{N_2,r_2,\mu_2}[n_2]$ from Table 2. We represent the term $A\delta(x - a)$ from f with a circle of radius proportional to $\sqrt{|A|}$ and center at a. When $A < 0$, we fill the circle.

3.3. **Periodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^3$.** We extend the analysis in the previous section, and analagously we generalize Theorem 3.1 in a three dimensional setting.

Theorem 3.3. *Let the generalized function* f *on* $\mathbb{R}^3$ *be an* a_1, a_2, a_3*-periodic eigenfunction of the Fourier transform operator* $\mathcal{F}$ *with eigenvalue* $\lambda = 1, -i, -1,$ *or* $+i$*. Assume that the linearly independent periods* a_1, a_2, a_3 *from* $\mathbb{R}^3$ *have been chosen as small as possible subject to the constraint that* $0 < \|a_1\| \le \|a_2\| \le \|a_3\|$*. Then there are positive integers* $N_1 \le N_2 \le N_3$ *such that*

$$\|a_1\| = \sqrt{N_1}, \quad \|a_2\| = \sqrt{N_2}, \quad \|a_3\| = \sqrt{N_3}$$

and there are nonnegative integers $0 \le M_1 < N_1, 0 \le M_2 < N_1, 0 \le M_3 < N_1 + N_2$ *such that* a_1*,*

$$a_2' := N_1 a_2 - M_1 a_1,$$

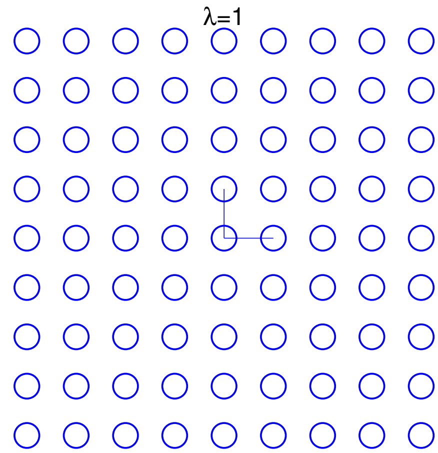

FIGURE 4. The $(1,0)^T, (0,1)^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda = 1$, constructed from $f_{1,0,1;1,0,1}$.

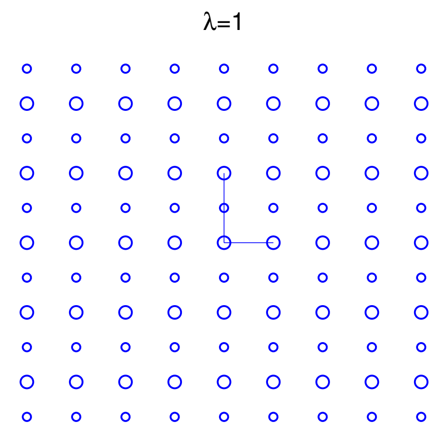

FIGURE 5. The $(1,0)^T, (0,\sqrt{2})^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda = 1$, constructed from $f_{1,0,1;2,0,1}$.

and

$$a_3^{'} := N_1[(M_1M_3 - N_2M_2)a_1 - (N_1M_3 - M_1M_2)a_2 + (N_1N_2 - M_1^2)a_3]$$

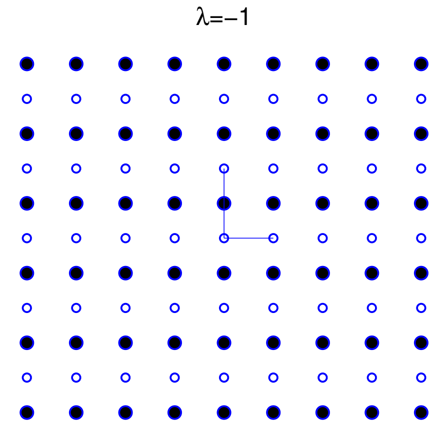

FIGURE 6. The $(1,0)^T, (0,\sqrt{2})^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda = -1$, constructed from $f_{1,0,1;2,2,1}$.

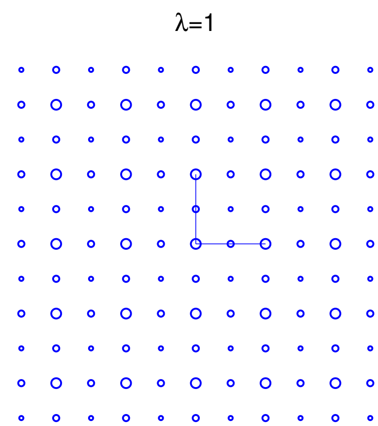

FIGURE 7. The $(\sqrt{2},0)^T, (0,\sqrt{2})^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda = 1$, constructed from $f_{2,0,1;2,0,1}$.

are pairwisely orthogonal with

$$\|a_2'\| = \sqrt{N_2'}, \qquad \|a_3'\| = \sqrt{N_3'}$$

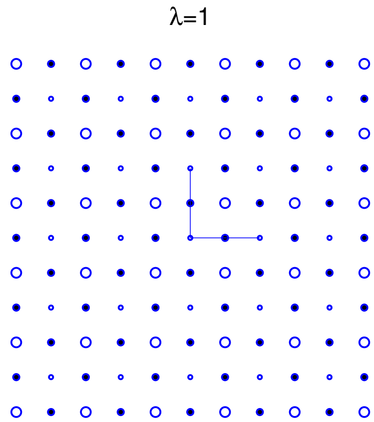

FIGURE 8. The $(\sqrt{2},0)^T,(0,\sqrt{2})^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda=1$, constructed from $f_{2,2,1;2,2,1}$.

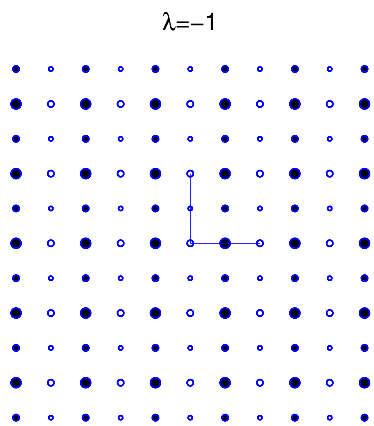

FIGURE 9. The $(\sqrt{2},0)^T,(0,\sqrt{2})^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda=-1$, constructed from $f_{2,0,1;2,2,1}$.

where

$$N_2^{'} := N_1(N_1N_2-M_1^2),$$

$$N_3^{'} := N_1^2(N_1N_2-M_1^2)[N_1N_2N_3+2M_1M_2M_3-(N_1M_3^2+N_2M_2^2+N_3M_1^2)].$$

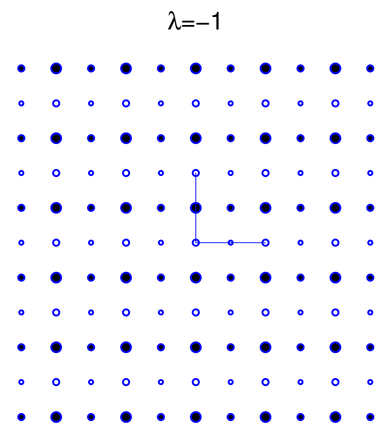

FIGURE 10. The $(\sqrt{2},0)^T,(0,\sqrt{2})^T$-periodic eigenfunction of the bivariate Fourier transform operator $\mathcal{F}$ with $\lambda=-1$, constructed from $f_{2,2,1;2,0,1}$.

The generalized function f is a_1, a_2', a_3'-periodic, and there is an orthogonal transformation Q such that

$$f_Q(x) := f(Qx)$$

is $(\sqrt{N_1},0,0)^T,(0,\sqrt{N_2'},0)^T,(0,0,\sqrt{N_3'})^T$-periodic with the representation

$$f_Q(x) = \sum_{m_1=-\infty,m_2=-\infty,m_3=-\infty}^{\infty} \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2'-1}\sum_{n_3=0}^{N_3'-1}\Big\{\gamma[n_1,n_2,n_3] \cdot\delta\Big(x_1-\frac{n_1}{\sqrt{N_1}}-m_1\sqrt{N_1}, x_2-\frac{n_2}{\sqrt{N_2'}}-m_2\sqrt{N_2'}, x_3-\frac{n_3}{\sqrt{N_3'}}-m_3\sqrt{N_3'}\Big)\Big\}. \tag{42}$$

Here

$$(\mathcal{F}_{N_1,N_2',N_3'}\ \gamma)[k_1,k_2,k_3] = \frac{1}{N_1N_2',N_3'}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2'-1}\sum_{n_3=0}^{N_3'-1}\Big\{\gamma[n_1,n_2,n_3] \cdot e^{-2\pi i(k_1n_1/N_1+k_2n_2/N_2'+k_3n_3/N_3')}\Big\} = \frac{\lambda}{\sqrt{N_1N_2'N_3'}}\gamma[k_1,k_2,k_3]$$

for $k_1=0,1,\dots,N_1-1,\ k_2=0,1,\dots,N_2'-1,\ k_3=0,1,\dots,N_3'-1.$

3.4. **Some quasiperiodic eigenfunctions of the Fourier transform operator on $\mathbb{R}^2$.** In this section we will construct some quasiperiodic eigenfunctions of the Fourier transform operator. A generalized function f is said to be quasiperiodic if the Fourier transform $f^\wedge$ is a weighted sum of Dirac δ functionals with isolated support [11, Definition 1.1, Definition 1.2, Definition 1.11, p. 27–31].

3.4.1. *The rotation theorem for grids on $\mathbb{R}^2$.*

Theorem 3.4. *Let a_1, a_2 be linearly independent vectors in $\mathbb{R}^2$ with*

$$|\det [a_1 \ \ a_2]| = 1.$$

Then

$$grid^\wedge_{a_1,a_2} = grid_{Qa_1,Qa_2}$$

where

$$Q := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{43}$$

is a 90° rotation.

Proof. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be chosen so that

$$\mathcal{A} := [a_1 \ \ a_2] = \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix}$$

has

$$\det \mathcal{A} = \alpha\gamma - \beta\delta = 1.$$

We compute

$$\mathcal{A}^{-T} = \det \mathcal{A} \begin{bmatrix} \gamma & -\delta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \gamma & -\delta \\ -\beta & \alpha \end{bmatrix},$$

$$Q\mathcal{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \delta & \gamma \end{bmatrix} = \begin{bmatrix} -\delta & -\gamma \\ \alpha & \beta \end{bmatrix},$$

and observe that the columns of these matrices generate the same lattice. □

Corollary 3.5. *Let a_1, a_2 be linearly independent vectors in $\mathbb{R}^2$ and assume that*

$$grid^\wedge_{a_1,a_2} = grid_{a_1,a_2}.$$

Then a_1, a_2 can be replaced by orthonormal vectors that generate the same grid.

Proof. From the support of grid_{a_1,a_2}, we choose a_1, a_2 to be as small as possible subject to $0 < \|a_1\| \le \|a_2\|$. Since

$$\text{grid}^\wedge_{a_1,a_2} = \text{grid}_{a_1,a_2},$$

we must have

$$|\det [a_1 \ \ a_2]| = 1.$$

This being the case, we must have

$$\text{grid}_{Qa_1,Qa_2} = \text{grid}^{\wedge}_{a_1,a_2} = \text{grid}_{a_1,a_2}.$$

It follows that Qa_1 is in the lattice generated by a_1, a_2, and since

$$\|Qa_1\| = \|a_1\|,$$

we must have

$$\|a_2\| = \|Qa_1\|.$$

If a_2 is distinct from $\pm Qa_1$, one of the vectors $a_1 \pm a_2$, $Qa_1 \pm a_2$ will be in the lattice and have a shorter length than $\|a_2\|$. Thus $a_2 = \pm Qa_1$. □

Corollary 3.6. *If*

$$|\det[a_1 \;\; a_2]| = 1,$$

and $grid_{a_1,a_2}$ *is distinct from* $grid^{\wedge}_{a_1,a_2}$*, then*

$$f_+(x) := grid_{a_1,a_2}(x) + grid^{\wedge}_{a_1,a_2}(x) \tag{44}$$

$$f_-(x) := grid_{a_1,a_2}(x) - grid^{\wedge}_{a_1,a_2}(x) \tag{45}$$

are eigenfunctions of the Fourier transform operator $\mathcal{F}$ *associated with* $\lambda = 1, \lambda = -1$*, respectively.*

Example 3.6. Using the biorthogonal system

$$a_1 = (2,0)^T, \quad a_2 = (0,1/2)^T, \quad A_1 = (1/2,0)^T, \quad A_2 = (0,2)^T$$

with

$$\det[a_1 \;\; a_2] = 1,$$

and we use (44) and (45) to obtain the bivariate eigenfunctions

$$f_+(x) = \text{grid}_{a_1,a_2}(x) + \text{grid}_{A_1,A_2}(x), \quad f_-(x) = \text{grid}_{a_1,a_2}(x) - \text{grid}_{A_1,A_2}(x)$$

of $\mathcal{F}$ as shown in Fig. 11 and Fig. 12. These eigenfunctions are both $(2,0)^T, (0,2)^T$-periodic on $\mathbb{R}^2$.

Example 3.7. Using the biorthogonal system

$$a_1 = (\alpha,0)^T, \quad a_2 = (0,1/\alpha)^T, \quad A_1 = (1/\alpha,0)^T, \quad A_2 = (0,\alpha)^T, \quad \alpha = \sqrt[4]{2},$$

with

$$\det[a_1 \;\; a_2] = 1,$$

and we use (44) and (45) to produce the eigenfunctions shown in Fig. 13 and Fig. 14, respectively. In this case neither of the eigenfunctions f_+ and f_- is periodic. They do have a discrete spectrum, however, so they are quasiperiodic.

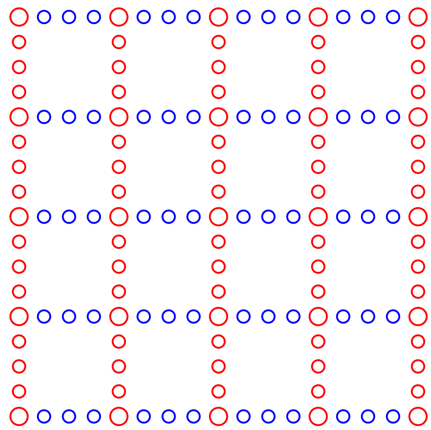

FIGURE 11. The periodic eigenfunction (44), of the bivariate Fourier transform $\mathcal{F}$ with $\lambda = 1$ constructed with $a_1 = (2, 0)^T$ and $a_2 = (0, 1/2)^T$.

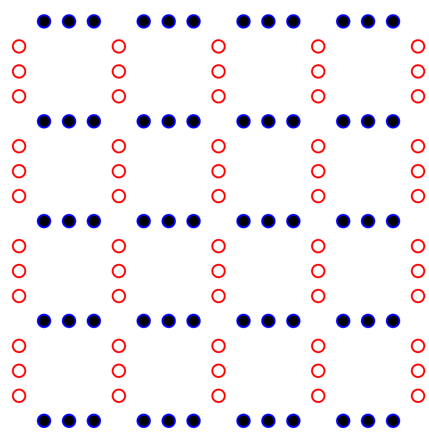

FIGURE 12. The periodic eigenfunction (45), of the bivariate Fourier transform $\mathcal{F}$ with $\lambda = -1$ constructed with $a_1 = (2, 0)^T$ and $a_2 = (0, 1/2)^T$.

3.4.2. *Quasiperiodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^2$ with m-fold rotational symmetry.* Let

$$\alpha = 1/\sqrt{\sin\left(\frac{2\pi}{n}\right)} \tag{46}$$

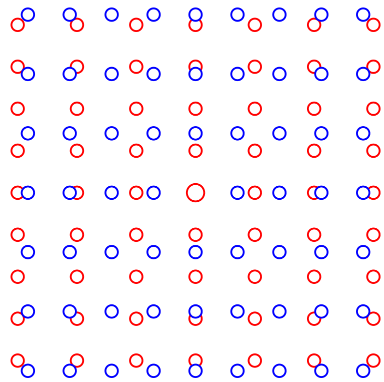

FIGURE 13. The quasiperiodic eigenfunction (44), of the bivariate Fourier transform $\mathcal{F}$ with $\lambda = 1$ constructed with $a_1 = (\alpha, 0)^T$ and $a_2 = (0, 1/\alpha)^T$ when $\alpha = \sqrt[4]{2}$.

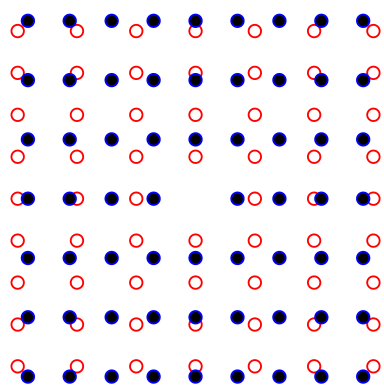

FIGURE 14. The quasiperiodic eigenfunction (45), of the bivariate Fourier transform $\mathcal{F}$ with $\lambda = -1$ constructed with $a_1 = (\alpha, 0)^T$ and $a_2 = (0, 1/\alpha)^T$ when $\alpha = \sqrt[4]{2}$.

for some $n = 3, 4, \ldots$, and let

$$a_k = \alpha\Big(\cos(2\pi k/n), \sin(2\pi k/n)\Big)^T, \quad k = 0, 1, \ldots, n-1, \tag{47}$$

be the vertices of a regular $n - gon$ with center at the origin. The parameter α has been chosen so that

$$\det\begin{bmatrix} a_k & a_{k+1} \end{bmatrix} = \alpha^2 \begin{bmatrix} \cos 2\pi k/n & \cos 2\pi(k+1)/n \\ \sin 2\pi k/n & \sin 2\pi(k+1)/n \end{bmatrix} = \alpha^2 \sin 2\pi/n = 1,$$

for each $k = 1, 2, \ldots, n-1$. In view of Theorem 3.4, this allows us to see that

$$\text{grid}^{\wedge}_{a_k,a_{k+1}} = \text{grid}_{Qa_k,Qa_{k+1}}, \qquad k = 0, 1, \ldots, n-1$$

(with $a_n := a_0$) where again

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is a quarter turn rotation. We will use this fact to generate quasiperiodic eigenfunctions of $\mathcal{F}$ on $\mathbb{R}^2$ with rotational symmetry.

Example 3.8. When $n = 3, \alpha = \sqrt{2/\sqrt{3}}$, so that

$$a_0 = \alpha(1,0)^T, \ a_1 = \alpha(-\frac{1}{2}, \frac{\sqrt{3}}{2})^T.$$

The corresponding hexagonal grid_{a_0,a_1} and

$$\text{grid}^{\wedge}_{a_0,a_1} = \text{grid}_{Qa_0,Qa_1}$$

are shown in Fig. 15. We note that grid_{a_0,a_1} has 6-fold rotational symmetry and that

$$\text{grid}_{a_0,a_1} = \text{grid}_{a_1,a_2} = \text{grid}_{a_2,a_3}.$$

We form

$$f_+ = \text{grid}_{a_0,a_1} + \text{grid}^{\wedge}_{a_0,a_1} \tag{48}$$

as shown in Fig. 16, noting that

$$f_+^{\wedge} = \text{grid}^{\wedge}_{a_0,a_1} + \text{grid}^{\wedge\wedge}_{a_0,a_1} = \text{grid}^{\wedge}_{a_0,a_1} + \text{grid}_{a_0,a_1} = f_+.$$

We also form

$$f_- := \text{grid}_{a_0,a_1} - \text{grid}_{a_0,a_1} \tag{49}$$

as shown in Fig. 17, noting that

$$f_-^{\wedge} = \text{grid}^{\wedge}_{a_0,a_1} - \text{grid}^{\wedge\wedge}_{a_0,a_1} = \text{grid}^{\wedge}_{a_0,a_1} - \text{grid}_{a_0,a_1} = -f_-.$$

Neither f_+ nor f_- is periodic. Indeed f_+ has the weighted spike $2\delta(x)$ at the origin. There will be another δ spike with weight 2 if and only if there are integers n_0, n_1, k_0, k_1 such that

$$n_0 a_0 + n_1 a_1 = k_0 Q a_0 + k_1 Q a_1,$$

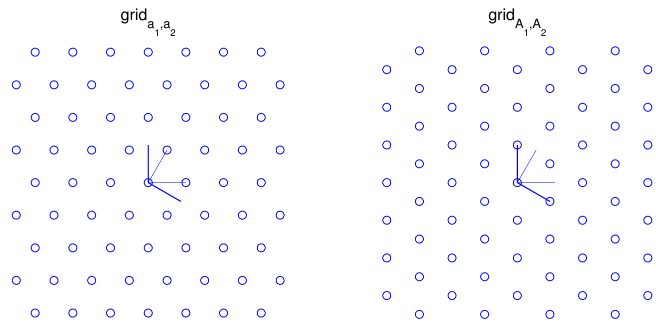

FIGURE 15. The grid_{a_1,a_2} and its dual grid_{A_1,A_2} from Example 3.8.

i.e.,

$$n_0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + n_1 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} = k_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + k_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix},$$
$$= k_0 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + k_1 \begin{bmatrix} -\sqrt{3} \\ -1 \end{bmatrix},$$

i.e.,

$$2n_0 - n_1 = -k_1\sqrt{3}$$
$$n_1\sqrt{3} = 2k_0 - k_1.$$

Since $\sqrt{3}$ is irrational we immediately conclude that

$$k_1 = n_1 = 0, \quad 2n_0 - n_1 = 2k_0 - k_1 = 0,$$

i.e.,

$$k_0 = k_1 = n_0 = n_1 = 0.$$

A similar argument applies to f_-.

We will now construct a family of quasiperiodic eigenfunctions of $\mathcal{F}$ that have rotational symmetry. Let $n = 3, 4, \ldots,$ and a_k, $k = 0, 1, 2, \ldots, n-1$ be given by (47), let α be given by (46), and let

$$f_{n+}(x) := \sum_{k=0}^{n-1} \text{grid}_{a_k,a_{k+1}}(x) + \text{grid}^{\wedge}_{a_k,a_{k+1}}(x), \tag{50}$$

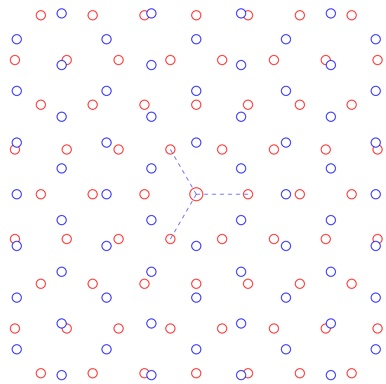

FIGURE 16. The eigenfunction f_{3+} as given by (48) (having 12-fold rotational symmetry) for the Fourier transform operator $\mathcal{F}$ with $\lambda = 1$.

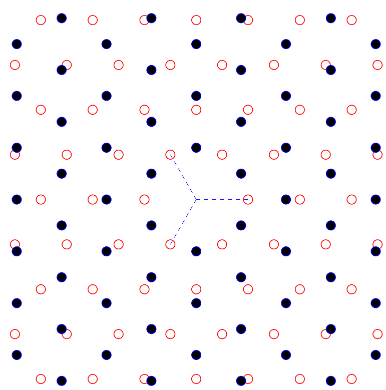

FIGURE 17. The eigenfunction f_{3-} as given by (49) (having 6-fold rotational symmetry) for the Fourier transform operator $\mathcal{F}$ with $\lambda = -1$.

and

$$f_{n-}(x) := \sum_{k=0}^{n-1} \mathrm{grid}_{a_k,a_{k+1}}(x) - \mathrm{grid}^{\wedge}_{a_k,a_{k+1}}(x), \tag{51}$$

(with $a_n := a_0$.) By construction,

$$f^{\wedge}_{n+} = f_{n+} \qquad \text{and} \qquad f^{\wedge}_{n-} = -f_{n-}.$$

When $n = 3$, f_{3+}, f_{3-} are scalar multiples of the functions f_+, f_- from (48),(49), respectively. When n is divisible by 4, Qa_k is included in the list $a_0, a_1, \ldots, a_{n-1}$ so

$$\sum_{k=0}^{n-1} \text{grid}^{\wedge}_{a_k,a_{k+1}} = \sum_{k=0}^{n-1} \text{grid}_{Qa_k,Qa_{k+1}} = \sum_{k=0}^{n-1} \text{grid}_{a_k,a_{k+1}}.$$

As a result, f_{n+} has n-fold rotational symmetry and

$$f_{n-} = 0.$$

Of course, f_{4+} is a scalar multiple of the grid on $\mathbb{Z} \times \mathbb{Z}$, see Fig. 18. Fig. 19 shows a represntation of f_{8+}.

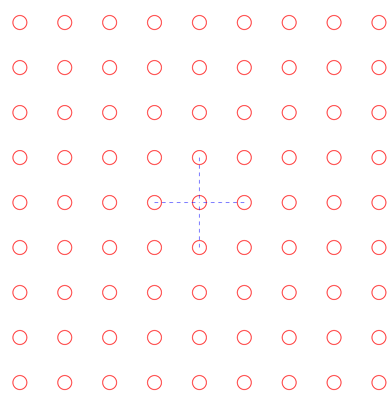

FIGURE 18. The periodic eigenfunction f_{4+} (having 4-fold rotational symmetry) as given by (50), for the Fourier transform operator $\mathcal{F}$ with $\lambda = 1$.

Example 3.9. Fig. 20–Fig. 23 show f_{n+}, f_{n-} for $n = 5, 7$. These generalized eigenfunctions of $\mathcal{F}$ are all quasiperiodic (but not periodic).

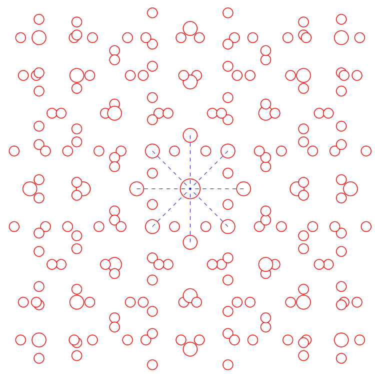

FIGURE 19. The quasiperiodic eigenfunction f_{8+} as given by (50) (having 8-fold rotational symmetry) for the Fourier transform operator $\mathcal{F}$ with $\lambda = 1$.

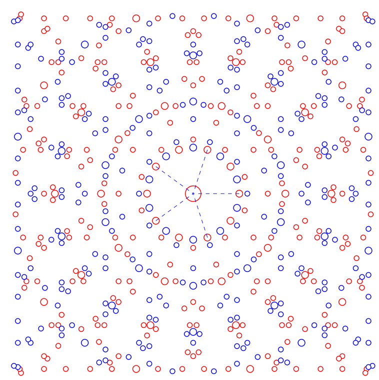

FIGURE 20. The quasiperiodic eigenfunction f_{5+} with 20-fold rotational symmetry for the Fourier transform operator $\mathcal{F}$ with $\lambda = 1$.

4. Conclusions

A generalization of Theorem 3.1 to dimensions higher than 3 is not known at this point, and will be the subject of a future work.

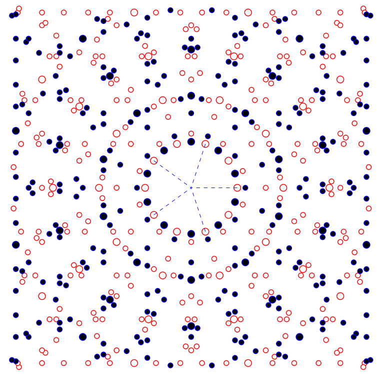

FIGURE 21. The quasiperiodic eigenfunction f_{5-} with 10-fold rotational symmetry for the Fourier transform operator $\mathcal{F}$ with $\lambda = -1$.

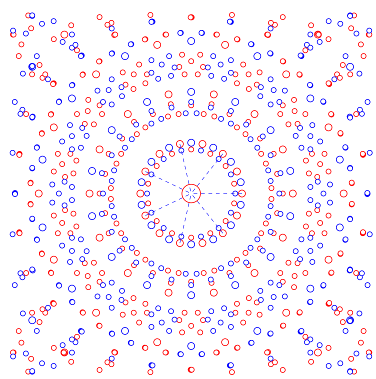

FIGURE 22. The quasiperiodic eigenfunction f_{7+} with 28-fold rotational symmetry for the Fourier transform operator $\mathcal{F}$ with $\lambda = 1$.

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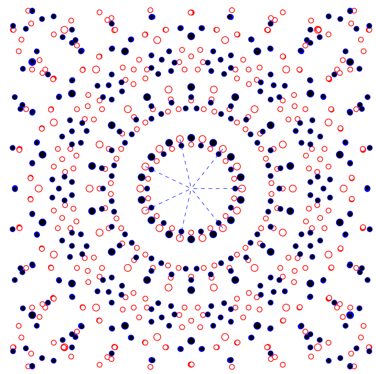

FIGURE 23. The quasiperiodic eigenfunction f_{7-} with 14-fold rotational symmetry for the Fourier transform operator $\mathcal{F}$ with $\lambda = -1$.

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