Using technology to teach and solve challenging math problems

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Abstract

Dr. Ellina Grigorieva is a Professor of Mathematics at Texas Woman's University (TWU). She graduated from Moscow State University first in her class of 300, defended her PhD in mathematics and physics, and has published over 50 scientific papers in the field of differential equations, economics and optimal control theory. She is a former USSR Math Olympiad winner, a member of the USAMO grading committee, and a panelist for AIME, ASHME and AMS.

For the past 25 years, whenever she spotted an especially interesting or tricky problem she added it to her notebooks, along with her original solutions. She has accumulated thousands of these problems, and she uses them in her everyday teaching. Her students are aware that her classes are unlike anything they have ever encountered. Her problems enable them to discern reusable patterns and techniques in problem solving. The author will share her ideas with other educators; she will show how problems involving numbers and proofs develop the intellectual skills of students. Starting from a simple example and moving to more complicated problems, you will learn how students in your calculus class can construct and investigate a microeconomic model and even predict the best production or sales strategy that maximizes profit. The workshop will be based on her own book "Complex math problems and how to solve them" and other sources of math contest problems, including Moscow State University entrance exam problems, USSR math Olympiad and USAMO contest problems. Methods will be presented for solving complex problems that use technology and recognition of patterns that extend from algebra.

In this talk, the author will teach how to use technology and pattern recognition in order to solve challenging problems from different areas of mathematics, such as calculus, geometry, algebra, and trigonometry.

1. Trajectories of a System

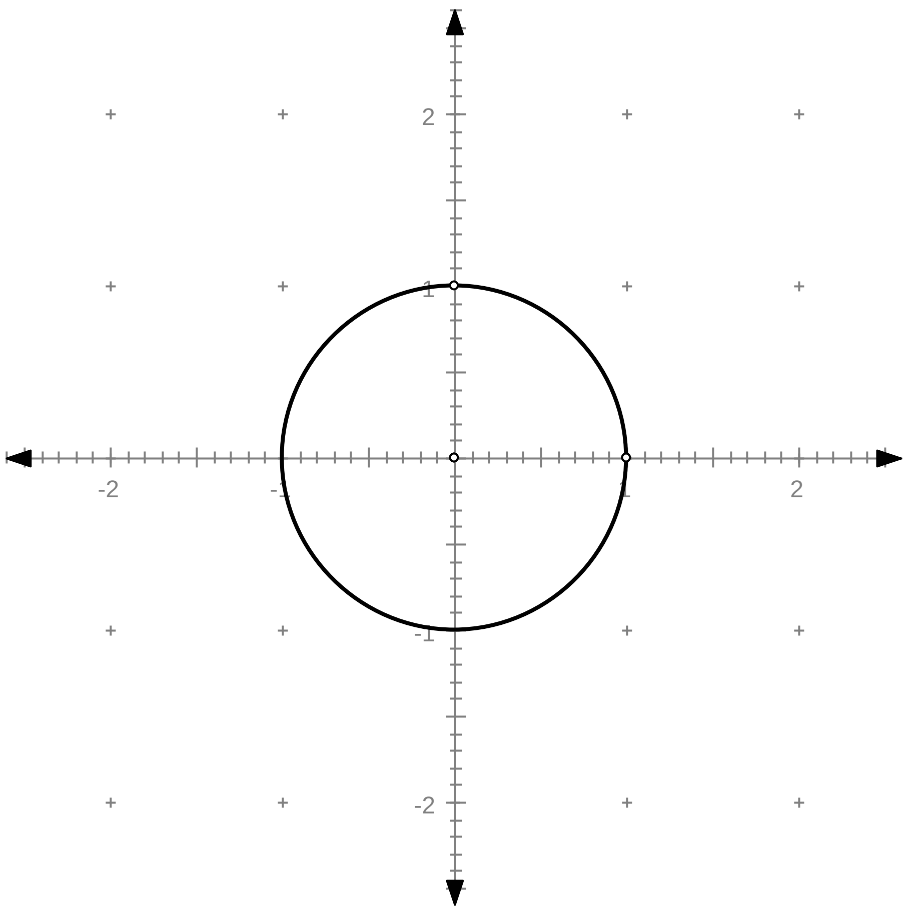
Let us consider the following system of two equations with a parameter

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \qquad \textbf{(1.1)}$$

What is the curve? Can you rewrite it as y=y(x)?

Squaring both sides and adding the two equations we obtain **a trajectory** of the system (1.1), a circle equation of radius 1, centered at (0,0):

$$x^2(t) + y^2(t) = 1$$



Ask your students which curve is described by the following system.

$$\begin{cases} x(t) = 4\cos t \\ y(t) = 3\sin t \end{cases} \qquad (1.\ 2)$$

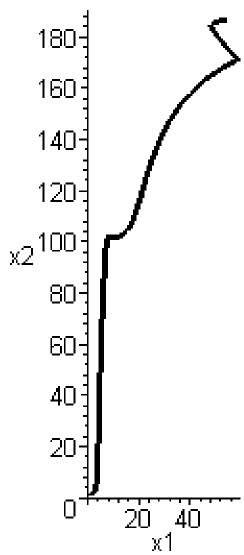
$$\begin{cases} \dfrac{x(t)}{4} = \cos t \\ \dfrac{y(t)}{3} = \sin t \end{cases} \qquad \frac{x^2}{(4)^2} + \frac{y^2}{(3)^2} = 1$$ This is an ellipse.

Ask students about the trajectory of this system:

$$\begin{cases} x(t) = 2\sin t \\ y(t) = 6\sin t + 5 \end{cases} \quad \textbf{(1. 3)}$$

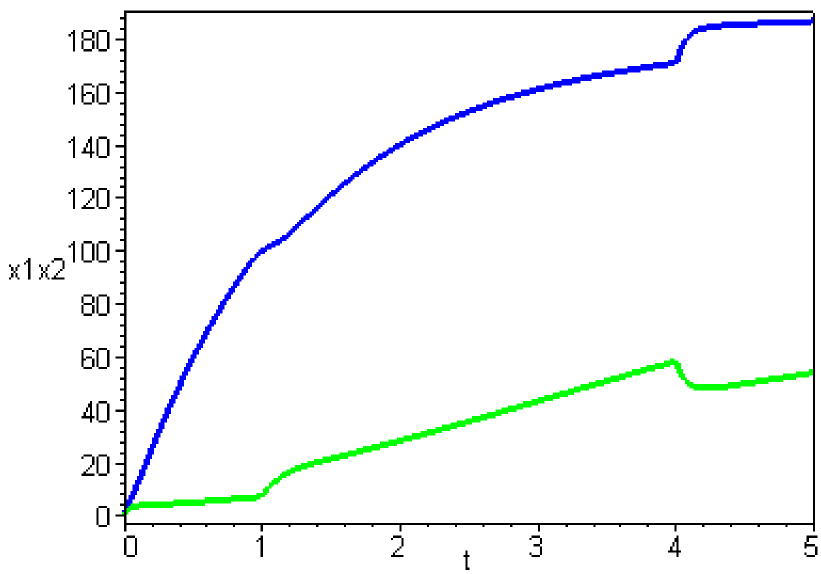
This would give you a line $y = 3x + 5$. You can graph it on a calculator and discuss why the graph looks like a segment, but not a line.
(Hint: Look at the range of x(t) : [-2, 2] and y(t): [-1, 11].) We use the **boundedness** of the function ***sin t***.

Equations with parameters help us to understand real world problems. For example, while modeling a microeconomic system of the process of production, storage and sales of some consumer good by a system of differential equations, and solving it numerically, we obtained the following trajectory describing $x_1(t)$ (the number of units of the good on the market) and $x_2(t)$ (the amount of the good sold but not consumed in consumers' homes). ([1])



Obtaining a graph like the one above, we do not specifically see the time, but this is **a trajectory** describing the behavior of our system on the time interval **[0, T]**. Every point on this trajectory can be visualized as an ordered pair **(** $x_1(t), x_2(t)$ **)**, and the end of the trajectory is the state at t=T (the end of the given time interval).

Modern technology allows us to graph $x_1 = x_1(t)$, $x_2 = x_2(t)$ as functions of time. Look at these trajectories. They are so different from the phase plane. Compare it with the straight line segment (system (1. 3)) and the graphs of x(t) and y(t) (sin curves).



We can learn how to understand a trajectory analytically by rewriting a parametric system of equations in a different form. For example, a system of differential equations **(1. 4)** can be solved in any AP calculus class:

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases} \qquad \textbf{(1.4)}$$

We find solutions as

$$x(t) = C_1 e^{-t} \ \text{ and } \ y(t) = C_2 e^{t}, \qquad \textbf{(1.5)}$$

where C1 and C2 are some constants, and ***t*** is an independent variable (time); so now we can eliminate time in **(1.5)** and rewrite the solution of the system **(1.4)** as a trajectory **y=y(x):**

$$y = \frac{K}{x}\ , \ \text{ where } K = C_1 \cdot C_2 \qquad \textbf{(1.6)}$$

Formula (6) tells us that solutions (trajectories) of system (4) are hyperbolas, which, depending on the value and sign of constant K, will be located in all four quadrants of the X-Y coordinate plane. On the other hand, we could graph trajectories on a TI 83 calculator in parametric MODE, giving different values to constants C1 and C2.

2. Methods of evaluation and estimation: Non-standard problems and problems with parameters

You can decide how necessary this material is for you after taking the quiz below. I give problems like these in order to see how deeply my students understand functions and their properties. Here they are, just three problems.

Find all real solutions of the following equations and inequality:

(2.1) $\frac{1}{x} + x = \sqrt{2}$

(2.2) $\sin(3x - 1) = 3$

(2.3). $2^x + 2^{-x} < \sqrt[5]{5}$

Did you get the solutions? Are you still working? Is it hard? Where did you get stuck?

Hint: Don't try to solve them, think about properties of the functions, and try to use common sense. None of three problems has a solution. Mathematicians say that their solutions are the empty set.

Now, look at each problem again, like you are solving it from the end. Can you explain why, for example, the first equation does not have a solution? Look at the left hand side of the equation, and assume first that **x>0**. Applying the relationship between the arithmetic and geometric means of two positive numbers ***a*** and ***b***

$$a + b \ge 2\sqrt{ab} \qquad (2.4)$$

and remembering that the equality appears if and only if a=b, we can see that

$$x + \frac{1}{x} \ge 2\sqrt{x \cdot \frac{1}{x}} = 2$$

So the left side of equation **(2.1)** is always greater than or equal to **2**, but the right side is a constant, $\sqrt{2}$. Because **2>** $\sqrt{2}$ **, equation (2.1) has no solutions for all positive x.** But you can say that maybe there is a solution when **x<0,** and that we can't apply formula (2.4) for negative numbers. Yes, when x<0 we cannot, but because –x>0, we can rewrite the left side as $x + \frac{1}{x} = -(-x - \frac{1}{x})$ and apply the familiar formula to the expression within parentheses.

For all x<0, $-x - \frac{1}{x} \ge 2$**, then** $x + \frac{1}{x} \le -2$

And again there is no solution for negative **x** because the left side is less than or equal to **–2**, but the right side is a constant $\sqrt{2}$. Therefore, equation **(2.1)** has no real solutions.

I guess you now know in what direction to go when looking for an explanation for the second problem. Maybe the second equation will be easy to understand. We want to compare the ranges of the functions on the left and on the right. If you have completely forgotten trigonometry, let's recall the definitions of sin and cosine functions.

Sin is **the y- coordinate** of a point on the unit circle (of radius 1)
Cosine is **the x- coordinate** of a point on the unit circle.
Any point on the unit circle can have coordinates only within interval [-1, 1]. We can conclude that the **sin** and **cosine** have the same range [-1, 1]; this can be written as $\mathbf{-1 \le \sin x \le 1 \text{ and } -1 \le \cos x \le 1}$.

Now, can you explain why equation **(2.2)** has no solution for real ***x***?
The left side is always less or equal to 1, but the right side is 3. This equation has no solutions.

What do you think about inequality **(2.3)?** It has no solutions as well. Can we prove it?
The left side is positive and the right is positive. Recalling properties of exponents, we notice that $\mathbf{2^x \text{ and } 2^{-x}}$ are reciprocals of each other. Now we can rewrite the left side as $\mathbf{2^x + \frac{1}{2^x}}$.

$$\mathbf{2^x + \frac{1}{2^x} \ge 2\sqrt{2^x \cdot \frac{1}{2^x}} = 2}$$

Since **2**> $\sqrt[5]{5}$ **,** in the inequality **(2. 3)** has no solutions**.**

We want to mention that all of these three problems have very understandable graphical interpretations: the functions on the left hand side have no points of intersection with functions on the right hand side. The purpose of this presentation is to show how to apply properties of functions and their boundaries to the solution of some non-standard problems, and problems with parameters. Those of you who solved all three problems without my help will find many new ideas and approaches you might enjoy and use in your classroom.

Problem 1

Solve the equation: $\mathbf{2^{1-|x|} = 1 + x^2 + \frac{1}{1+x^2}}$

Solution

You can notice that the left side of the equation is always positive and decreasing in ***x***. It is less than or equal to **2** for any real ***x***. The expression on the right is positive as well and

$$1+x^2+\frac{1}{1+x^2}\geq 2\sqrt{(1+x^2)\frac{1}{(1+x^2)}} = 2$$

(We again used the inequality between **geometric and arithmetic means**)
The solution can appear only when both sides equal 2. Thus, x=0 is the solution.

Answer: x=0

Problem 2

Let f(x)= $\sqrt{x^2-4x+4}-3$ and g(x)= $\sqrt{x}-a$. Solve the inequality $f(g(x))\leq 0$ for x.

Solution

Completing the square under the radical of the first function and using properties of an absolute value, we can rewrite **f(x)** as $\mathbf{f(x)=\sqrt{(x-2)^2}-3=|x-2|-3}$ ($\mathbf{g(x)=\sqrt{x}-a}$)

Next, we find the composition of **f(x)** and **g(x), *(*f $\circ$ g*):***

$$\mathbf{f(g(x))=|(\sqrt{x}-a)-2|-3} \qquad \mathbf{(1)}$$

By the condition of the problem expression **(1)** must be less than or equal to zero.

$$f(g(x))\leq 0 \Leftrightarrow |(\sqrt{x}-a)-2|-3\leq 0 \Leftrightarrow |\sqrt{x}-a-2|\leq 3 \Leftrightarrow -3\leq\sqrt{x}-a-2\leq 3$$

$$\mathbf{a-1\leq\sqrt{x}\leq a+5} \qquad \mathbf{(2)}$$

Let us consider three cases for **(2):**

1. If $\mathbf{a+5<0}$ (a<-5), then we have no solutions.
2. If $\mathbf{a-1<0\leq a+5}$ $\mathbf{(-5<a<1)}$, **then** $\mathbf{\sqrt{x}\leq a+5\Leftrightarrow 0\leq x\leq(a+5)^2}$.
3. If $\mathbf{0\leq a-1}$ $\mathbf{(a\geq 1)}$, **then** $\mathbf{a-1\leq\sqrt{x}\leq a+5\Leftrightarrow(a-1)^2\leq x\leq(a+5)^2}$.

<u>Answer</u>: no solutions for a<-5; $\mathbf{x\in[0,(a+5)^2]}$ ***if –5<a<1;*** $\mathbf{x\in[(a-1)^2,(a+5)^2]}$ ***if***
***a*≥ 1.**

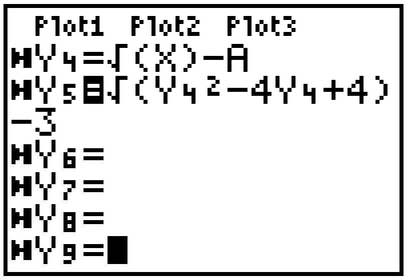
Having solved this problem in general, for any value of a parameter ***a***, you can always find some particular solution of the problem, and answer a question like: If **a=10**, what ***x*** will satisfy the given inequality $\mathbf{f(g(x))\leq 0}$?

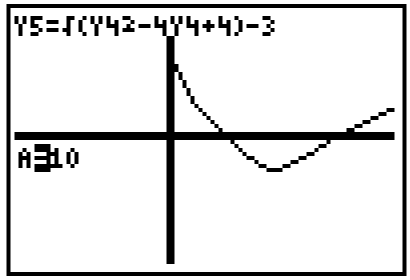
Analytically we obtained that if $\boldsymbol{a\geq 1}$, then $\boldsymbol{x\in[(a-1)^2,(a+5)^2]}$.

If this is true, then this must be true for **a=10** as well, and the graph of function **y=f (g(x))** must go below the X-axis only for $\boldsymbol{x\in[(10-1)^2,(10+5)^2]}$ or $\boldsymbol{81\leq x\leq 225}$.

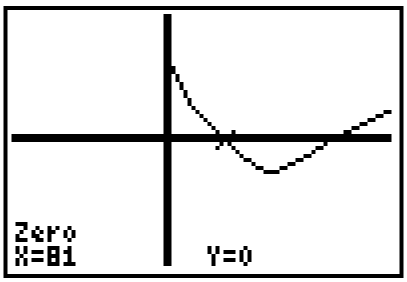
Let us check the answer on a TI 83/84 plus graphing calculator.

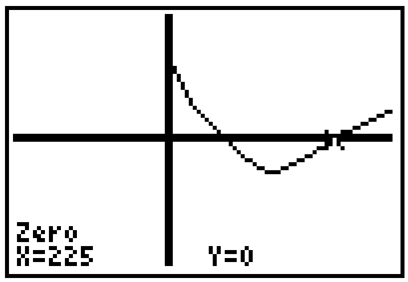
First we make a composition of two functions on our graphing calculator. Here $Y_4 = g(x)$ and $Y_5 = f(g(x))$.





We expect "negative" behavior between **x=81** and **x=225**, then we need to set up an appropriate window for our calculator. Next we use **[2nd] [TRACE] (CALC)** buttons to find zeros of **f(g(x))**:





As we expected, they are precisely 81 and 225.

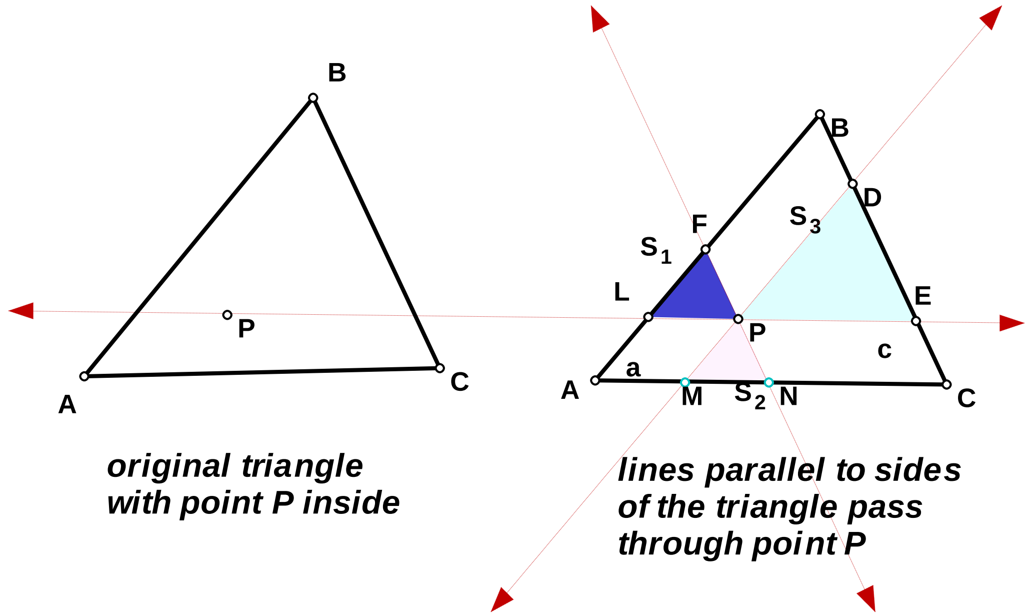
3. Some Geometry Challenge

Similar Triangles: You need to remember that two triangles are similar to each other by two sides, by two angles (AA), by two sides and the included angle (SAS), and by three sides (SSS). Especially important is the fact that in similar triangles the ratio of corresponding sides, medians, heights, and bisectors equals k, the ***coefficient of similitude***. The ratio of the areas of similar triangles equals k^2, the square of the coefficient of similitude. The following problem will illustrate how I teach plane geometry with the use of technology.

Triangle ABC is given. Point P is interior with respect to the triangle. Through this point P three lines parallel to each side of the triangle are drawn, and form three triangles with areas $S_1, S_2, and\ S_3$. Find the area, S, of triangle ABC.

Solution

In order to even start thinking about this problem I recommend drawing a picture.



Having a picture, we notice that LE//AC, MD//AB, and FN//BC, then we can conclude that LP=AM and PE=NC. It seems reasonable to denote three variables: **a=AM, b=MN, and c=NC.** Besides, each of the small triangles is similar to triangle ABC. How can we use this information? Of course, you remember the ratio of areas of similar triangles equals a square of the coefficient of similitude, $\boldsymbol{k}^2$. However, how can we find these three coefficients? Again we know the answer to this question: the coefficient of similitude, ***k***, is the ratio of corresponding sides of similar triangles. Now we are ready to solve the problem.

$$\Delta FLP \approx \Delta ABC \Rightarrow \frac{S_1}{S} = \left(\frac{LP}{AC}\right)^2$$

$$\Delta MPN \approx \Delta ABC \Rightarrow \frac{S_2}{S} = \left(\frac{MN}{AC}\right)^2$$

$$\Delta PDE \approx \Delta ABC \Rightarrow \frac{S_3}{S} = \left(\frac{PE}{AC}\right)^2$$

Replacing **AC=a + b + c, LP=a, MN=b,** and **PE=c** and taking the square root of the left and the right sides of each equality, we have

$$\frac{\sqrt{S_1}}{\sqrt{S}} = \frac{a}{a+b+c}$$
$$\frac{\sqrt{S_2}}{\sqrt{S}} = \frac{b}{a+b+c}$$
$$\frac{\sqrt{S_3}}{\sqrt{S}} = \frac{c}{a+b+c}$$

Does it look too far from the answer? Try to add the left and the right sides.

Putting expressions over the common denominator, we obtain:

$\frac{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}}{\sqrt{S}} = \frac{a+b+c}{a+b+c} = 1$ that is equivalent to $\sqrt{S} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}$

Squaring both sides of the last equation gives us the answer:

$$S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$$

Answer: $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$.

References

[1] Grigorieva E. V. and E.N. Khailov, **2005**. "An Attainable Set of a Nonlinear Controlled Microeconomic Model". ***Journal of Nonlinear Dynamics and Control***, Vol. 11, No. 2, 157-176.