A HIERARCHICAL DIFFERENTIAL GAME BETWEEN A MANUFACTURER AND THE STATE IN AN ECOLOGICAL SETTING

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1. Description of the Model

Consider a manufacturer producing a single product, which is always in demand, on a given time interval. Its production activity is the cause of pollution. The manufacturer and the state control the levels of pollution.

Let q be the cost of production funds of the manufacturer, and $F(q)$ is the volume of the production. Assuming that it sells everything that it produces at the market price p, let p_0 be the unit price of the produced good, and $p > p_0$. Then, $pF(q)$ is the corresponding sales revenue of the manufacturer. Part of it, in the amount of $p(1-u)F(q)$ will be saved and the other portion $puF(q)$ will be invested into cleaning the environment. Here u and $(1-u)$ give the actual portion of each investment. Moreover, we assume that on a given time interval the cost of production funds (q) is unchanged.

Let s be the pollution stock from the production activity of the manufacturer. On one hand, this pollution stock will increase proportionally to the volume of production $rF(q)$. On the other hand, it will decrease with the rate of natural pollution degradation σs and additionally as a result of the investments of the manufacturer and the state into environmental cleaning with the rates $\beta puF(q)$ and ρvs, respectively. Here r, σ, β, ρ are coefficients of proportionality.

Therefore, the dynamics of pollution stock $s(t)$ on the given time interval $[0, T]$ will be described by the following Cauchy problem:

$$\begin{cases} \dot{s}(t) = rF(q) - (\sigma + \beta p u(t) F(q) + \rho v) s(t),\ t \in [0, T], \\ s(0) = s_0; s_0 > 0. \end{cases} \tag{1.1}$$

The right side of the differential equation from (1.1) contains the control function $u(t)$, satisfying the inequalities:

$$0 \le u(t) \le 1,\ t \in [0, T], \tag{1.2}$$

and also the control parameter v, satisfying the restrictions:

$$0 \le v \le v_+. \tag{1.3}$$

For the manufacturer and the state we consider the corresponding objective functions, which define the cumulative profits for the given time interval:

$$J_M(u) = \int_0^T \Big((1 - u(t)) p F(q) - p_0 F(q) - \lambda g(c, s(t)) \Big) dt, \tag{1.4}$$

$$J_S(v, c) = \int_0^T \Big(\chi p F(q) + \lambda g(c, s(t)) - v s(t) \Big) dt. \tag{1.5}$$

In the functional (1.4) the first term describes the amount of the sales revenue; the second term is the production expense; the third term determines the penalty (fine) paid by the manufacturer to the state if it exceeds the maximum level of pollution c. It is assumed that

$$g(c, s) = 0.5 \cdot (\max\{0; s - c\})^2,$$

and λ is the coefficient of proportionality. Here the value c is the control parameter, satisfying the restrictions:

$$0 < c_- \le c \le c_+. \tag{1.6}$$

In the functional (1.5) the first term sets the manufacturer's tax on profit; the second and third terms are determined above.

Now, we define the control set $D(T)$ as the set of all Lebesgue measurable functions $u(t)$, satisfying the inequalities (1.2).

For the Cauchy problem (1.1) we formulate **The Main Problem** as follows: for the manufacturer to choose such control $u(\dot{}) \in D(T)$, that maximizes the objective function $J_M(u)$; for the state to choose such control parameters v, c, satisfying the restrictions (1.3),(1.6), that maximize the objective function $J_S(v, c)$.

Similar to reality it is considered that the state possesses reliable information about the possible behavior of the manufacturer on the given time interval. It gives some advantage to the state. Therefore, the established problem is considered as a hierarchical differential game, in which the state is a leader. Hence, it follows from [1] that **The Main**

Problem can be reduced to the consecutive solution of the following particular problems:

Problem 1. At given control parameters v, c, satisfying the inequalities (1.3),(1.6), for the Cauchy problem (1.1) find the control $u(\cdot) = u(\cdot, v, c) \in D(T)$, maximizing the objective function $J_M(u)$.

Problem 2. Using the solution to Problem 1, i.e. the function $u(t) = u(t, v, c)$, for the Cauchy problem (1.1) find such control parameters $v = v(u(\cdot, v, c))$, $c = c(u(\cdot, v, c))$, satisfying the restrictions (1.3),(1.6), for which the objective function $J_S(v, c)$ is maximized.

For analysis of these problems we will use the following property of the value s.

Lemma 1. *Let any control $u(\cdot) \in D(T)$ and control parameter v, satisfying the inequalities* (1.3)*, be given. Then the solution $s(t)$ of the equation from* (1.1) *satisfies the restriction*

$$s(t) > 0,\ t \in [0, T].$$

Let us introduce the following values:

$$\alpha = rF(q),\ \gamma = pF(q). \tag{1.7}$$

Then, the Cauchy problem (1.1) and functionals (1.4),(1.5) can be rewritten as:

$$\begin{cases} \dot{s}(t) = \alpha - (\sigma + \beta\gamma u(t) + \rho v)s(t),\ t \in [0, T], \\ s(0) = s_0,\ s_0 > 0, \end{cases} \tag{1.8}$$

$$J_M(u) = (p - p_0)F(q)T + I_M(u),\ J_S(v, c) = \chi\gamma T + I_S(v, c), \tag{1.9}$$

where

$$I_M(u) = -\int_0^T (\gamma u(t) + \lambda g(c, s(t)))dt, \tag{1.10}$$

$$I_S(v, c) = \int_0^T (\lambda g(c, s(t)) - vs(t))dt. \tag{1.11}$$

From the formulas (1.7),(1.9) it follows that subsequent arguments of the considered differential game are executed for the Cauchy problem (1.8) and the objective functions (1.10),(1.11).

2. Analysis of Problem 1

Let some control parameters v, c, satisfying the inequalities (1.3),(1.6), be given. Then, the existence of the optimal solution for Problem 1 - the optimal control $u_*(t)$ and its corresponding optimal trajectory $s_*(t)$, $t \in [0, T]$ follows from [2]

In order to solve Problem 1 we will apply the Pontryagin Maximum Principle [3]. It means that for the optimal control $u_*(t)$ and its corresponding optimal trajectory $s_*(t)$ there exists a non-trivial solution $\psi_*(t)$ to the Cauchy problem

$$\begin{cases} \dot{\psi}_*(t) = (\sigma + \beta\gamma u_*(t) + \rho v)\psi_*(t) + \lambda \dot{g}(c, s_*(t)),\ t \in [0, T], \\ \psi_*(T) = 0, \end{cases} \tag{2.1}$$

for which the following relationship is valid

$$u_*(t) = \begin{cases} 0 & , \ \text{if } \ \eta_*(t) < 0, \\ \forall u \in [0, 1] & , \ \text{if } \ \eta_*(t) = 0, \\ 1 & , \ \text{if } \ \eta_*(t) > 0, \end{cases} \tag{2.2}$$

where

$$\eta_*(t) = -\beta s_*(t)\psi_*(t) - 1. \tag{2.3}$$

The function $\eta_*(t)$ is the switching function. We can see from (2.2) that it's behavior determines the type of optimal control $u_*(t)$.

Cauchy problems (1.8),(2.1) and relationships (2.2),(2.3) form a two point boundary value problem for the Maximum Principle. We will study this problem in depth. If the control $u(t)$ and the trajectory $s(t)$ with the function $\psi(t)$ satisfy this boundary value problem, then $u(t)$ is called the extremal control, $s(t)$ is the extremal trajectory, and $\psi(t)$ is its corresponding solution of the Cauchy problem (2.1). Moreover, we consider that $\eta(t)$ is the corresponding switching function. Since the Pontryagin Maximum Principle is only the necessary condition of optimality, then the optimal control $u_*(t)$ and the optimal trajectory $s_*(t)$ with its corresponding function $\psi_*(t)$ satisfy the boundary value problem of the Maximum Principle. Therefore, it is necessary for us to study the properties of the extremal control $u(t)$, and the extremal trajectory $s(t)$ with its corresponding function $\psi(t)$.

We have the following property of the extremal control $u(t)$:

Lemma 2. *There exists such moment $\tau \in [0, T)$, such that on the interval $(\tau, T]$ the extremal control $u(t)$ takes value 0.*

Next, we will consider for $s > 0$ the following nonlinear equation

$$\frac{\alpha}{\lambda\beta s^2} = \dot{g}(c, s).$$

It is easy to see that this equation has a unique solution $\widehat{s}$, satisfying the restriction $\widehat{s} > c$.

Then, for $s > 0$ we will introduce the auxiliary function

$$w(s) = \frac{\alpha}{\beta\gamma s} - \frac{\sigma + \rho v}{\beta\gamma},$$

which is a decreasing function. For this function the following relationships are valid:

$$\lim_{s\to+0} w(s) = +\infty,\ \lim_{s\to+\infty} w(s) = -\frac{\sigma+\rho v}{\beta\gamma}.$$

Using this function we find the value

$$\widehat{u} = w(\widehat{s}). \tag{2.4}$$

We have the following property of the extremal control $u(t)$:

Lemma 3. *The extremal control $u(t)$ is a piecewise constant function taking values $\{0; \widehat{u}; 1\}$ if $\widehat{u} \in (0,1)$, and values $\{0; 1\}$ if $\widehat{u} \notin (0,1)$.*

From this statement it follows that the relationship (2.2) for the extremal control $u(t)$ can be rewritten as

$$u(t) = \begin{cases} 0 & \text{, if } \eta(t) < 0, \\ \begin{cases} 0 & \text{, if } \widehat{u} \le 0, \\ \widehat{u} & \text{, if } 0 < \widehat{u} < 1 \\ 1 & \text{, if } \widehat{u} \ge 1, \end{cases} & \text{, if } \eta(t) = 0, \\ 1 & \text{, if } \eta(t) > 0. \end{cases} \tag{2.5}$$

Now, using the equations from (1.8),(2.1) and the formula (2.3) we will obtain the Cauchy problem for the switching function $\eta(t)$ of the type

$$\begin{cases} \dot{\eta}(t) = \frac{\alpha}{s(t)}\eta(t) + \left(\frac{\alpha}{s(t)} - \lambda\beta s(t)\dot{g}(c, s(t))\right),\ t \in [0,T], \\ \eta(T) = -1. \end{cases} \tag{2.6}$$

Then, using the function $w(s)$ we will define the values $s_{u=0}$, $s_{u=1}$ by the following equalities:

$$w(s_{u=0}) = 0,\ w(s_{u=1}) = 1. \tag{2.7}$$

Values $s_{u=0}$, $s_{u=1}$ are the equilibrium points of the differential equation from (1.8) under the controls $u = 0$ and $u = 1$, respectively. From the decreasing nature of the function $w(s)$ and equalities (2.4),(2.7) we can state that the following inequalities are true:

$$\left[\begin{array}{ll} s_{u=1} < s_{u=0} \le \widehat{s} & \text{, if } \widehat{u} \le 0, \\ s_{u=1} < \widehat{s} < s_{u=0} & \text{, if } 0 < \widehat{u} < 1, \\ \widehat{s} \le s_{u=1} < s_{u=0} & \text{, if } \widehat{u} \ge 1. \end{array}\right. \tag{2.8}$$

Using the Cauchy problems (1.8),(2.6) and the relationship (2.5) the two point boundary value problem for the Maximum Principle can be rewritten as

$$\begin{cases} \dot{s}(t) = \alpha - (\sigma + \beta\gamma u(s(t),\eta(t)) + \rho v)s(t),\ t \in [0,T], \\ \dot{\eta}(t) = \frac{\alpha}{s(t)}\eta(t) + \left(\frac{\alpha}{s(t)} - \lambda\beta s(t)\dot{g}(c,s(t))\right), \\ s(0) = s_0,\ \eta(T) = -1,\ s_0 > 0, \end{cases} \tag{2.9}$$

where the control $u(s,\eta)$ is redefined at $\eta = 0$ with the inequalities (2.8) and the function $w(s)$ in the following way

$$u(s,\eta) = \begin{cases} 0 & ,\ \text{if}\ \ \eta < 0, \\ \begin{cases} 0 & ,\ \text{if}\ \ s \geq s_{u=0}, \\ w(s) & ,\ \text{if}\ \ s_{u=1} < s < s_{u=0} \\ 1 & ,\ \text{if}\ \ s \leq s_{u=1}, \end{cases} & ,\ \text{if}\ \ \eta = 0, \\ 1 & ,\ \text{if}\ \ \eta > 0. \end{cases} \tag{2.10}$$

Now, for the boundary value problem (2.9),(2.10) we will consider its corresponding Hamiltonian system

$$\begin{cases} \dot{s}(t) = \alpha - (\sigma + \beta\gamma u(s(t),\eta(t)) + \rho v)s(t), \\ \dot{\eta}(t) = \frac{\alpha}{s(t)}\eta(t) + \left(\frac{\alpha}{s(t)} - \lambda\beta s(t)\dot{g}(c,s(t))\right), \end{cases} \tag{2.11}$$

where the function $u(s,\eta)$ is defined by the formula (2.10).

The system of equations (2.10),(2.11) is an example of the so-called piecewise "cross-linked" systems [4]. Above the line $\eta = 0$ all trajectories of the system (2.10),(2.11) are trajectories corresponding to the control $u(s,\eta) = 1$. Below the line $\eta = 0$ trajectories correspond to the control $u(s,\eta) = 0$. The line $\eta = 0$ by itself is the line of "cross-linking", but not a trajectory of the system (2.10),(2.11). Moreover, trajectories coming close to the line $\eta = 0$ either go through it, changing the value of control $u(s,\eta)$ from 0 into 1, or from 1 into 0, or stay tangent to this line. Detailed analysis of the Hamiltonian system (2.10),(2.11) is presented in [5]. Similar investigations of the Hamiltonian systems related to economic growth problems are presented in [6]-[9].

Based on this analysis we have the following results. At first, we will consider the case, when $\widehat{u} \leq 0$ or $\widehat{u} = 1$. The statement below is valid.

Theorem 1. *In Problem 1 the optimal control $u_*(t)$ is either a constant function of the type*

$$u_*(t) = 0,\ t \in [0,T]; \tag{2.12}$$

or a piecewise constant function with one switching of the type

$$u_*(t) = \begin{cases} 1 & ,\ \textit{if}\ \ 0 \leq t < \tau, \\ 0 & ,\ \textit{if}\ \ \tau \leq t \leq T, \end{cases} \tag{2.13}$$

where $\tau \in (0,T)$ is the moment of switching.

Next, we will consider the case, when $0 < \widehat{u} < 1$. The following statement is valid.

Theorem 2. *In Problem 1 the optimal control* $u_*(t)$ *is either a constant function of the type* (2.12)*;*
or a piecewise constant function with one switching of the type (2.13)*;*
or a piecewise constant function with one switching of the type

$$u_*(t) = \begin{cases} \widehat{u} & , \text{ if } \ 0 \le t < \tau, \\ 0 & , \text{ if } \ \tau \le t \le T, \end{cases} \tag{2.14}$$

where $\tau \in (0, T)$ *is the moment of switching;*
or a piecewise constant function with two switchings of one of the types:

$$u_*(t) = \begin{cases} 0 & , \text{ if } \ 0 \le t < \tau_1, \\ \widehat{u} & , \text{ if } \ \tau_1 \le t \le \tau_2, \\ 0 & , \text{ if } \ \tau_2 < t \le T, \end{cases} \tag{2.15}$$

$$u_*(t) = \begin{cases} 1 & , \text{ if } \ 0 \le t < \tau_1, \\ \widehat{u} & , \text{ if } \ \tau_1 \le t \le \tau_2, \\ 0 & , \text{ if } \ \tau_2 < t \le T, \end{cases} \tag{2.16}$$

where $\tau_1, \tau_2 \in (0, T)$ *are the moments of switching.*

At last, we will consider the case, when $\widehat{u} > 1$. The following statement is true.

Theorem 3. *In Problem 1 the optimal control* $u_*(t)$ *is either a constant function of the type* (2.12)*;*
or a piecewise constant function with one switching of the type (2.13)*;*
or a piecewise constant function with two switchings of the type

$$u_*(t) = \begin{cases} 0 & , \text{ if } \ 0 \le t < \tau_1, \\ 1 & , \text{ if } \ \tau_1 \le t \le \tau_2, \\ 0 & , \text{ if } \ \tau_2 < t \le T, \end{cases} \tag{2.17}$$

where $\tau_1, \tau_2 \in (0, T)$ *are the moments of switching.*

In this way, at the given control parameters v, c, satisfying the inequalities (1.3),(1.6), the optimal control $u_*(t)$ depending on the value $\widehat{u}$ has one of the types (2.12)-(2.17).

3. Solution to Problem 1

Now, we will describe the method of solution to Problem 1. Let us introduce the set Λ, which for $\widehat{u} \in (0,1)$ is defined by the relationship

$$\Lambda = \Big\{\theta = (\theta_1, \theta_2, \theta_3) \in \mathrm{R}^3 : 0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq T\Big\},$$

and for $\widehat{u} \notin (0,1)$ is given by the formula

$$\Lambda = \Big\{\theta = (\theta_1, \theta_2) \in \mathrm{R}^2 : 0 \leq \theta_1 \leq \theta_2 \leq T\Big\}.$$

Then, for any point $\theta \in \Lambda$ we define the control $u_\theta(t)$ of the type

$$u_\theta(t) = \begin{cases} 0, \text{ if } & 0 \leq t \leq \theta_1, \\ 1, \text{ if } & \theta_1 < t \leq \theta_2, \\ \widehat{u}, \text{ if } & \theta_2 < t \leq \theta_3, \\ 0, \text{ if } & \theta_3 < t \leq T \end{cases}$$

if $\widehat{u} \in (0,1)$, and of the type

$$u_\theta(t) = \begin{cases} 0, \text{ if } & 0 \leq t \leq \theta_1, \\ 1, \text{ if } & \theta_1 < t \leq \theta_2, \\ 0, \text{ if } & \theta_2 < t \leq T, \end{cases}$$

if $\widehat{u} \notin (0,1)$. It is easy to see that the control $u_\theta(t)$ includes all possible types (2.12)-(2.17) of the optimal control $u_*(t)$ at the corresponding values of switchings θ_i.

Next, we substitute the control $u_\theta(t)$ into the differential equation from (1.8) and integrate it on the interval $[0,T]$. Then, we substitute the corresponding function $s_\theta(t)$ into the functional $J_M(u)$. Hence, we have the function of three variables

$$F(\theta_1, \theta_2, \theta_3) = J(u_\theta), \ (\theta_1, \theta_2, \theta_3) \in \Lambda,$$

if $\widehat{u} \in (0,1)$, and the function of two variables

$$F(\theta_1, \theta_2) = J(u_\theta), \ (\theta_1, \theta_2) \in \Lambda,$$

if $\widehat{u} \notin (0,1)$.

Therefore, Problem 1 can be restated as a problem of the finite dimensional optimization of the type

$$F(\theta_1, \theta_2, \theta_3) \to \min_{(\theta_1,\theta_2,\theta_3)\in\Lambda}, \tag{3.1}$$

if $\widehat{u} \in (0,1)$, and of the type

$$F(\theta_1, \theta_2) \to \min_{(\theta_1,\theta_2)\in\Lambda}, \tag{3.2}$$

if $\widehat{u} \notin (0,1)$. The methods for the numerical solution of the problems (3.1),(3.2) are well developed [10],[11].

4. Solution to Problem 2

Now, let us consider Problem 2 with the known function $u_*(t)$, $t \in [0,T]$. Since we solved Problem 1 approximately, then we will find an analogous solution for Problem 2. For this, on the rectangle $[0, v_+] \times [c_-, c_+]$, in which the control parameters v, c change, we will set the uniform grid (v_i, c_j), $i = \overline{0,N}$, $j = \overline{0,M}$ with steps $h_v = \frac{v_+}{N}$, $h_c = \frac{c_+ - c_-}{M}$. Then, we have equalities:

$$v_i = ih_v, \ i = \overline{0,N}; \ c_j = c_- + jh_c, \ j = \overline{0,M}.$$

At every node (v_i, c_j) of such a grid we find the corresponding function $u_*^{ij}(t)$, $t \in [0,T]$, which is the solution of Problem 1. For values v_i, $u_*^{ij}(t)$, $t \in [0,T]$ we find the solution $s_*^{ij}(t)$ of the differential equation from (1.8). After that, we define the value of the functional $J_S^{ij} = J_S(v_i, c_j)$.

Based on these calculations we find values i_0, j_0, for which the value $J_S^{i_0 j_0}$ is the maximum among all the values J_S^{ij}, $i = \overline{0,N}$, $j = \overline{0,M}$. Then, its corresponding control parameters v_{i_0}, c_{j_0} form the approximate solution of Problem 2.

5. Solution of the Main Problem

From the previous arguments we see that the control parameters v_{i_0}, c_{j_0} supply the maximum value of the objective function $J_S(v,c)$. Moreover, the corresponding control $u_*^{i_0 j_0}(t)$, $t \in [0,T]$ simultaneously gives the maximum value of the objective function $J_M(u)$. In this way, the values v_{i_0}, c_{j_0}, $u_*^{i_0 j_0}(t)$, $t \in [0,T]$ form the approximate solution of the Main Problem.

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