On The Erlanger Programm of Felix Klein

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1. Abstract

Here we present the philosophy of the Erlanger Programm, a tool to introduce and describe modern geometries. At the core of this philosophy is the geometric concept of *congruence* or *geometric transformation*, which turns out to make possible the uniform development and comparison of different geometries. Non-Euclidean geometries are hardly learned and explored by our students when presented in the axiomatic format. Though synthetic methods should not be neglected, the use of analytical methods is more appropriate to introduce and investigate various geometries. Thus is imperative for anyone who takes this approach to have a good understanding of the dual algebraic-geometric nature of complex numbers, which is in fact, the essence of analytic geometry. *Möbius transformations* must be studied carefully in order to understand the action of a *group* of such transformations, which constitutes the *engine* of a geometry defined through the Erlanger Programm. Lastly, for the most elegant computation processes in these geometries, integration methods from multivariable calculus must be known well. For example, *distance* and *area* in both hyperbolic and elliptic geometries can be defined via line integrals and double integrals respectively.

2. Problem and Solution (a glimpse of the power of *Möbius transformations*).

One day I asked my students to find a bijection that will put the unit circle over the line

$$x + y = 3$$

They came back with the *stereographic projection mapping* from the center of projection, the point $P(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ which was coupled with to the *point-at-infinity* of the complex plane. The work was quiet extensive but not too difficult though. However if we want to *cut* the unit circle at a point other than the one above, then the finding of the corresponding mapping could become tedious. A rotation might be needed besides the stereographic projection to achieve that. Or, if one is interested in overlapping the line in a particular way, for example, to have the point $B(0,1)$ of the unit circle sent over the point $N(0,3)$ of the line, $A(1,0)$ over $M(3,0)$, and $C(-1,0)$ sent to the point-at-infinity, then the problem becomes difficult to be solved unless a more sophisticated tool is being used. Here is a solution to the new problem:

"According to the *Fundamental Theorem of Möbius Geometry*, there is a unique Möbius transformation T that will send i to $3i$, 1 to 3, and -1 to ∞. Since the *cross-ratio* is *invariant* in Möbius Geometry, we get $(Tz, 3i, 3, \infty) = (z, i, 1, -1)$, and from here

$$Tz = \frac{(6+3i)z - 3i}{z+1}, |z| = 1\text{"}$$

This seems to be amazing and disappointing at the same time! Amazing because this so rapidly found mapping does what was asked for (check!) and disappointing because the work above uses mathematical language that is usually unheard in our classrooms. It is precisely the mathematical language within the Erlanger Programm that once learned will open the door to a new and more powerful kind of mathematical dialog.

3. Pythagorean Theorem is *strictly Euclidean* (a glimpse of the power of the *distance* formula in the disk model of hyperbolic geometry).

It is quite amazing to realize, for example, that the Pythagorean Theorem is a *strictly Euclidean* statement. In other words we can prove that *any proof* of it requires in some form the use of the Euclidean parallel postulate! And one way to arrive to this realization is through the elegant distance formula of the Poincare disk model of the hyperbolic geometry.

Note: Statements like the one above are priorities of the Erlanger Programm.

Here is the distance formula in the disk model of hyperbolic geometry:

Definition *In the hyperbolic plane, the **length** of a smooth curve γ with parametrization z(t) = x(t) + iy(t) (where a ≤ t ≤ b) is given by*

$$\ell(\gamma) = 2\int_a^b \frac{|z'(t)|\,dt}{1-|z(t)|^2}, \text{ where } z'(t) = x'(t)) + iy'(t)$$

Let z_1 and z_2 be two points in the hyperbolic plane. The distance from z_1 to z_2, written $d(z_1, z_2)$, is defined to be the length of the hyperbolic straight-line segment between the two points.

A simple proof of the **Pythagorean Theorem** presented in most books relies on two pairs of similar triangles (determined by the main altitude of a right triangle), the existence of which implies Euclidean geometry. But this doesn't guarantee that there is no other proof that makes no use of the parallel postulate. To solve this dilemma we examine whether the Pythagorean theorem is independent of the first four postulates. If that's going to be the case, then we may logically say that it is impossible to deduce the Pythagorean theorem from the first four axioms and conclude that it strictly Euclidean. All we need is a geometric model that satisfies the first four axioms and fails the fifth. In such a model then we need to find a right triangle failing the Pythagorean theorem. Yes, we do have the disk model of the hyperbolic geometry for this task, and using the distance formula we can show immediately that the triangle with vertices $O(0,0)$, $A(-.6, 0)$, $B(0,.5)$ does not satisfy the Pythagorean theorem. It follows that the Pythagorean theorem is strictly Euclidean.

4. **The Main Frame**

Congruence, born from the group of transformations, tells us what's worth of studying in a given geometry. All these geometric entities worth of consideration are called **invariants**. Geometries are classified by the way they defined congruence.

Congruence comes first and any kind of ***measurement*** is worth of studying only if it is proved in advance that it remains unchanged when applied to congruent figures.

The group of transformations is the ***engine*** of the geometry. It allows us a **certain** freedom of motion. The larger the group, the more power is there to move figures around to create multiple representations of the same abstract geometric concept, thus lesser invariant sets. The smaller the group, the more invariants the geometry has, and the more difficult to study it.

The group of all *Möbius transformations* as given by

$$w = Tz = \frac{az+b}{cz+d}$$

where a, b, c, and d are complex constants, and the quantity ad - bc (the determinant of T) is not zero,

gives rise to the largest geometry over the extended complex plane, called the *Möbius Geometry*. It contains as subgeometries altogether, *Euclidean*, *Hyperbolic*, and *Elliptic* geometries. Whatever is invariant in *Möbius Geometry* will remain invariant in all these subgeometries.

The amazing power of analytical methods offers us, the intrigued explorers, direct access to the amazing beauties and wonders of these strange lands.