REMARKS ON STRONG REGULAR REFLECTION FOR THE ISENTROPIC GAS DYNAMICS EQUATIONS

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Abstract: We consider a two-dimensional Riemann problem for the isentropic gas dynamics equations that models regular shock reflection with a subsonic state immediately behind the reflected shock. The problem is rewritten using self-similar coordinates and the one-dimensional discontinuities in the far field are resolved using the one-dimensional theory of hyperbolic conservation laws. We obtain a system that changes type (elliptic, hyperbolic) and a free boundary problem that describes the subsonic state and the reflected shock. We conveniently rewrite the problem again and we obtain a second order equation for density with mixed boundary conditions, two first order equations for pseudo-velocities with Dirichlet boundary conditions, and an ordinary differential equation that describes the position of the reflected shock. We present summary of the proof of existence of a solution to this free boundary problem using the theory of second order elliptic equations and fixed point theorems.

1. Introduction. We consider two-dimensional isentropic gas dynamics equations:

$$\begin{aligned} &\rho_t + (\rho u)_x + (\rho v)_y = 0, \\ &(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0, \\ &(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = 0, \end{aligned} \tag{1}$$

where ρ stands for density, u and v denote velocities in x- and y-directions, respectively, and $p = p(\rho)$ is pressure. We denote $c^2(\rho) := p'(\rho)$ and assume that c^2 is a positive and increasing function of $\rho > 0$.

We consider initial data that is constant in two sectors. More precisely, the initial data is given by (Figure 1)

$$\overline{U}(x,y,0) = \begin{cases} \overline{U}_1 = (\rho_1, 0, 0), & x > sign(y)ky, \\ \overline{U}_0 = (\rho_0, u_0, 0), & \text{otherwise.} \end{cases} \tag{2}$$

We fix densities $\rho_0 > \rho_1 > 0$ arbitrarily. We further fix $k > 0$ large enough and we choose u_0 depending on ρ_0, ρ_1 and k, so that the following conditions are satisfied:

- both initial discontinuities $x = \pm ky$, $x > 0$ result in shocks (called *incident shocks*) followed by linear waves,

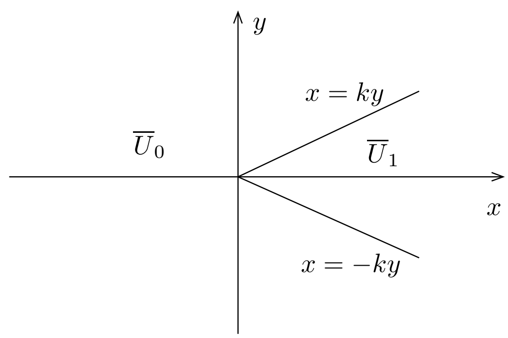

Figure 1: Riemann initial condition

- these two incident shocks interact at ξ-axis (at the reflection point Ξ_s),
- the quasi-one-dimensional Riemann problem at the reflection point Ξ_s results in two shocks (called *reflected shocks*) and an intermediate state between them, and
- the reflection point Ξ_s is subsonic with respect to this intermediate state.

We remark that there are two possible intermediate states for the quasi-one-dimensional Riemann problem at Ξ_s which results in two shocks. One of these two states is always subsonic with respect to the reflection point Ξ_s, while the other intermediate state is, for some choices of k, supersonic and, for other choices, subsonic. In this paper, we restrict our attention to the conditions when the intermediate state is subsonic, so that the resulting solution models strong (or transonic) regular reflection. More details on the choice of initial data could be found in [4] by Jegdić.

In this paper, we present summary of the proof of existence of a local solution to the two-dimensional Riemann problem (1)-(2). Our main tools are

- theory of second order elliptic equations with mixed (oblique and Dirichlet) boundary conditions, developed by Gilbarg, Trudinger and Lieberman in [3, 8]-[13], and
- various fixed point arguments.

This work follows earlier studies of strong regular reflection for the unsteady transonic small disturbance equation in [1], by Čanić, Keyfitz and Kim, and the nonlinear wave system in [6], by Jegdić, Keyfitz and Čanić.

2. Formulation of the problem. We follow the general approach outlined by Keyfitz in [7], and we rewrite the above system (1) using the self-similar variables $\xi = x/t$ and $\eta = y/t$. We obtain the following mixed type system in (ξ, η)-plane

$$\begin{aligned} &(\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ &(U, V) \cdot \nabla U + U + p_\xi/\rho = 0, \\ &(U, V) \cdot \nabla V + V + p_\eta/\rho = 0, \end{aligned} \tag{3}$$

where $U := u - \xi$ and $V := v - \eta$ denote pseudo-velocities.

When linearized about a constant state $\overline{U}_* = (\rho_*, u_*, v_*)$, this system is hyperbolic outside of the circle

$$(u_* - \xi)^2 + (v_* - \eta)^2 = c^2(\rho_*),$$

and mixed (hyperbolic-elliptic) inside. Note that the type of the system depends both on the location in (ξ, η)-plane as well on the solution $\overline{U} = (u, v, \rho)$ at that point.

We find a solution in the far field using the standard theory of one-dimensional hyperbolic conservation laws and we obtain a free boundary problem for the subsonic state and the reflected shocks. More details about the derivation of this free boundary problem could be found in [4] by Jegdić.

Since the problem is symmetric about ξ-axis, we restrict our attention to the upper half-plane. Furthermore, we restrict our analysis to a region close to the reflection point by introducing a cut-off boundary σ (see Figure 2) along which we impose an appropriate Dirichlet condition for density.

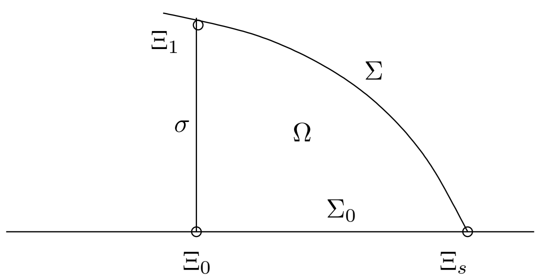

Figure 2: Domain Ω and its boundary

The free boundary problem is given by

- system (3) for ρ, U and V in the domain Ω,
- Rankine-Hugoniot jump conditions for ρ, U and V along the reflected shock Σ in the upper half-plane,
- symmetry conditions for ρ, U and V along the symmetry boundary Σ_0,
- Dirichlet conditions for ρ, U and V at the reflection point Ξ_s which are obtained by solving a quasi-one-dimensional Riemann problem at Ξ_s, and
- Dirichlet condition for ρ along the cut-off boundary σ.

We remark that the problem is a free boundary problem as the position of the reflected shock Σ depends (via Rankine-Hugoniot jump conditions) on the states on both sides of the shock and the solution inside Ω is unknown.

To prove existence of a solution to this free boundary problem, we rewrite system (3), conveniently, using a second order equation for density ρ and two first order equations for pseudo-velocities U and V, as is given in (4).

Moreover, we rewrite the jump conditions along Σ using an oblique derivative boundary condition for ρ, Dirichlet conditions for U and V and the first order ordinary differential equation that describes the position of Σ. These conditions are given in (5).

Hence, the free boundary problem that we study is of the form

$$\left.\begin{array}{l} a_{ij}(\rho,U,V)D^{ij}\rho + b_i(\rho,U,V)D^i\rho + c_{ij}(\rho,U,V)D^i\rho D^j\rho = 0 \\ (U,V)\cdot\nabla U + U + p_\xi/\rho = 0 \\ (U,V)\cdot\nabla V + V + p_\eta/\rho = 0 \end{array}\right\} \text{ in } \Omega \tag{4}$$

$$\left.\begin{array}{l} \beta(\rho,U,V)\cdot\nabla\rho = F(\rho,U,V) \\ U = G(\rho,\xi,\eta) \\ V = H(\rho,\xi,\eta) \\ \frac{d\eta}{d\xi} = \Psi(\rho,\xi,\eta) \end{array}\right\} \text{ on } \Sigma : \eta = \eta(\xi) \tag{5}$$

$$\rho_\eta = U_\eta = V = 0 \text{ on } \Sigma_0 \tag{6}$$

$$\rho = f \text{ on } \sigma \tag{7}$$

$$\overline{U}(\Xi_s) = \overline{U}_F, \qquad \eta(\xi_s) = 0, \tag{8}$$

where $\overline{U}_F$ is the subsonic intermediate state found by solving the quasi-one-dimensional Riemann problem at the point Ξ_s.

The theory of second order elliptic equations with mixed boundary conditions developed by Gilbarg, Trudinger and Lieberman in [3, 8]-[13] applies when the second order equation is uniformly elliptic and the oblique derivative boundary condition is uniformly oblique. To ensure that

- the second order equation for ρ (first equation in (4)) is uniformly elliptic,
- the oblique derivative boundary condition for ρ (first condition in (5)) is uniformly oblique, and that
- the shock evolution equation (forth equation in (5)) is well-defined,

we introduce various cut-off functions in the coefficients on these equations and conditions. Later, we show that these cut-off functions can be removed in a domain close to the reflection point Ξ_s by using the known solution $\overline{U}_F$ at the point Ξ_s.

3. Proof of existence. To prove existence of a solution to the free boundary problem (4)-(8), we proceed in three steps:

- we fix a free boundary Σ within a certain space of admissible curves,
- we solve the fixed boundary problem for ρ, U and V, and
- we update the location of Σ using the first order equation for the reflected shock.

The most challenging part of the proof is the second step – proving existence of a solution of the fixed boundary problem. Note that this fixed boundary problem is nonlinear. Moreover, we are not able to decouple the problem into a problem for density and a problem for pseudo-velocities as it was possible in the study of the nonlinear wave system in [6]. Therefore, we needed to develop novel techniques involving various fixed point arguments.

As the problem is nonlinear, we start with linearization of the problem in the following way. We fix ω, W and Z in a certain weighted Holder space $H_{1+\epsilon}^{(-\gamma)}$ where $0 < \epsilon < \gamma < 1$. More about Holder spaces and weighted Holder spaces could be found in [3] by Gilbarg & Trudinger.

We first linearize the second order equation for density (first equation in (4)) using ω, W and Z in the coefficients a_{ij}, b_i and c_{ij} and we replace the term $D^i\rho$, in the quadratic part of the equation, by $D^i\omega$. We consider this second order linear equation in Ω with a linearized oblique derivative boundary condition on Σ (the first condition in (5)) where we use ω, W and Z in β and F, together with symmetry condition (6) on Σ_0, and Dirichlet conditions (7) on σ and (8) at Ξ_s. Using the ideas from earlier studies on the steady/unsteady transonic small disturbance equations and the nonlinear wave system (see [7] by Keyfitz), we show that there exists a solution $\rho \in H_{1+\alpha}^{(-\gamma)}$ of this linear second order problem, where $0 < \epsilon < \alpha < \gamma < 1$. We remark that the solution ρ is "smoother" than ω (since $\alpha > \epsilon$). Using a fixed point argument, we show that the map

$$\omega \mapsto \rho$$

has a fixed point $\rho[W, Z] \in H_{1+\alpha}^{(-\gamma)}$. We emphasize that the solution ρ depends on functions W and Z.

Next, we linearize the first order equations for U and V (second and third equations in (4)) using W, Z and $\rho[W, Z]$ in the coefficients. We consider these equations in domain Ω together with Dirichlet conditions along Σ (see (5)), symmetry conditions on Σ_0 (see (6)) and Dirichlet conditions (8) at Ξ_s. We show that there exist a solution $U, V \in H_{1+\epsilon}^{(-\gamma)}$ of this linear problem. Since U and V are in the same space as W and Z, we use Banach contraction principle to show that the map

$$(W, Z) \mapsto (U, V)$$

is a contraction in a domain close to Ξ_s, where the size of this smaller domain depends only on initial data. Hence, this map has a fixed point $U, V \in H_{1+\epsilon}^{(-\gamma)}$. Therefore, $\rho[U, V]$, U and V solve the fixed boundary problem.

Once, we have a solution ρ, U and V of the fixed boundary problem (with fixed boundary curve Σ), we update the position of this boundary using the first order ordinary differential equation (fourth equation in (5)) with initial condition in (8). In the ordinary differential equation for the location $\eta(\xi)$, we use the value of ρ found as a solution of the fixed boundary problem with boundary Σ. This gives a new position $\tilde{\Sigma}$ and we prove that the map

$$\Sigma \mapsto \tilde{\Sigma}$$

on the space of admissible boundaries $H_{1+\alpha}$ has a fixed point.

Finally, we show that the cut-off functions introduced earlier could be removed in a region close to the reflection point.

This completes the proof of existence of a local solution to the nonlinear free boundary problem (4)-(8).

4. Conclusion. We study a Riemann problem for the two-dimensional isentropic gas dynamics equations that results in regular shock reflection with a subsonic state behind the reflected shock. This type of reflection is known as strong (or transonic) regular reflection. We present summary of the proof of existence of a local solution to this problem by rewriting it as a free boundary problem and by using the theory of second order elliptic equations with mixed boundary conditions, theory of hyperbolic equations and various fixed point arguments.

The problem where the state immediately behind the reflected shock is supersonic with respect to the reflection point Ξ_s models weak (or supersonic) regular reflection. Existence of a solution to this problem was proved for the unsteady transonic small disturbance equation in [2] by Čanić, Keyfitz and Kim, and for the nonlinear wave system in [5] by Jegdić. We believe that those ideas could be generalized and that one could also prove existence of a solution to a weak regular reflection problem for the isentropic gas dynamics equations.

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