Generalized Matrix Graphs and Completely Independent Critical Cliques

John J. Lattanzio and Quan Zheng

Department of Mathematics,
Indiana University of Pennsylvania, Indiana, PA 15705
John.Lattanzio@iup.edu
Quan.Zheng@iup.edu

Abstract

A k-dimensional n-square matrix is defined and certain properties of such matrices are investigated. Two particular graph constructions involving k-dimensional n-square matrices are given and the graphs so constructed are called matrix graphs. Properties of matrix graphs are determined and an application of matrix graphs to completely independent critical clique is provided. Some attention is given to this application and its relationship with the double-critical conjecture that the only vertex double-critical graph is the complete graph.

Keywords and phrases: matrix graph, chromatic number, critical clique, k-matching, completely independent critical cliques, double-critical conjecture
Running Head: Generalized matrix graphs
AMS Subject Classification: 05C15

1 Introduction and notation

In the late 1940's, G. A. Dirac defined critical graphs for the purpose of simplifying the central problems in the theory of graph coloring. Several results on critical graphs containing few edges have been established; e.g., [4], [5], [10], and [11]. Likewise, several results on critical graphs containing many edges have been established; e.g., [15] and [7]. However, determining bounds on the number of edges in critical graphs is not the objective of this paper. Rather, relations between particular sets of critical vertices will be investigated. This paper is a continuation and generalization of the results obtained in [12].

Most of the notation and terminology follows that found in [2] and [3]. The graphs considered in this paper are finite, undirected, and simple. For a given graph G, the vertex set of G and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order of G is the cardinality of $V(G)$ and is denoted by $|V(G)|$. The complete graph having order r shall be denoted by K_r. An r-clique of G is a subgraph K of G isomorphic to K_r. A subset I of $V(G)$ is said to be independent whenever no two distinct vertices in I are adjacent. The maximum cardinality of an independent subset of $V(G)$ is denoted by $\alpha(G)$. A subset M of $E(G)$ is said to be independent whenever no two edges in M share a common vertex; an independent subset M of $E(G)$ is often referred to as a matching. A matching M is called a k-matching whenever $|M| = k$. For a subset X of $V(G)$, the subgraph of G induced by X is denoted by $G[X]$. All vertex colorings considered will be proper; i.e., a partition of $V(G)$ into independent subsets of $V(G)$ called color classes. Lastly, the minimum cardinality of a partition of $V(G)$ admitted by a proper vertex coloring of G is called the chromatic number of G and is denoted by $\chi(G)$.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of natural numbers. For $n \in \mathbb{N}$, the set S_n denotes the set of all permutations on n elements. The general element σ in S_n will be written as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. When convenient, σ may also be written as a formal string, $\sigma = \sigma_1\sigma_2\cdots\sigma_n$. If $\tau \in S_n$ with $\tau = \tau_1\tau_2\cdots\tau_n$, then $\widehat{\tau}(i)$ denotes the formal substring of τ determined by deleting the ith character of τ. More precisely,

$$\widehat{\tau}(i) = \begin{cases} \tau_2\tau_3\cdots\tau_n & \text{, if } i = 1 \\ \tau_1\cdots\tau_{i-1}\tau_{i+1}\cdots\tau_n & \text{, if } 1 < i < n \\ \tau_1\tau_2\cdots\tau_{n-1} & \text{, if } i = n. \end{cases}$$

The jth character of the formal string $\widehat{\tau}(i)$, for $1 \le j \le n-1$, will be denoted by $\widehat{\tau}_j(i)$ and is given by

$$\widehat{\tau}_j(i) = \begin{cases} \tau_j & \text{, if } j < i \\ \tau_{j+1} & \text{, if } j \ge i. \end{cases}$$

Let A and B be sets. The symmetric difference of A and B, denoted by $A \otimes B$, is the set $(A - B) \cup (B - A)$. Let $\mathcal{S} = \{S_\omega : \omega \in \Omega\}$ be an indexed family of sets. The generalized union of the indexed family is denoted by $\cup\mathcal{S}$ and is given by

$$\cup\mathcal{S} = \bigcup_{\omega\in\Omega} S_\omega.$$

For $n \in \mathbb{N}$, an arbitrary Latin square of order n will be denoted by L_n. The ith row of a general Latin square L_n, where $1 \le i \le n$, will be denoted by $\Lambda_i = \lambda_{i1}\lambda_{i2}\cdots\lambda_{in}$. As usual, δ_{ij} will denote the Kronecker delta function. Lastly, let

$$M = \{a_1 \cdot x_1,\, a_2 \cdot x_2,\, \ldots,\, a_m \cdot x_m\}$$

be a [finite] multiset having m distinct elements $x_1, x_2, \ldots, x_m$. Here, the natural numbers $a_1, a_2, \ldots, a_m$ are called the repetition numbers and denote the number of times the corresponding element appears in the multiset M. Hence, it is clear that

$$|M| = \sum_{i=1}^{m} a_i.$$

A submultiset of M is a set

$$T = \{s_1 \cdot x_1,\, s_2 \cdot x_2,\, \ldots,\, s_m \cdot x_m\}$$

satisfying $0 \le s_i \le a_i$ for $i \in I_m$. An r-submultiset satisfies

$$|T| = \sum_{i=1}^{m} s_i = r$$

and may also be referred to as an r-combination.

2 Generalized k-dimensional n-square matrices

2.1 Terminology and basic definitions

For $n \in \mathbb{N}$, let I_n represent the nth segment of $\mathbb{N}$, that is, $I_n = \{1, 2, \ldots, n\}$. Also, for $k, n \in \mathbb{N}$, let I_n^k be the k-fold Cartesian product of I_n:

$$I_n^k = \prod_{i=1}^{k} I_n = I_n \times I_n \times \cdots \times I_n\,.$$

The general element of I_n^k will be denoted by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$. When convenient, α may also be viewed as a function, $\alpha : I_k \longrightarrow I_n$, defined by the rule $\alpha(i) = \alpha_i$, and written as a formal string $\alpha = \alpha_1\alpha_2 \cdots \alpha_k$. Moreover, we adopt the conventions that $I_n^0 = \{1\}$ and $I_n^1 = I_n$. For $T \subseteq \mathbb{N}$ and $c \in \mathbb{N}$, the scalar product cT is defined by $cT = \{ct : t \in T\}$.

Definition 1 *Let S be any nonempty set. A k-dimensional n-square matrix A over S is any function $A : I_n^k \to S$. Let $\mathcal{M}_n^k$ be the set of all k-dimensional n-square matrices*

For the purposes of this paper, the set S of Definition 1 will play no role. Therefore, specific mention of S will be omitted and we will simply say that A is a k-dimensional n-square matrix. Furthermore, Definition 1 imposes no conditions on k and n. However, we will require, except for one instance in Section 4, that $n \geq k$. The usual matrix notation will be adopted. That is, the general entry of $A \in \mathcal{M}_n^k$ will be denoted by a_α, where $\alpha \in I_n^k$. Also, A will be written as $A = [a_\alpha]$. For $A \in \mathcal{M}_n^k$, let A^* be the underlying set consisting of the entries of A; i.e., $A^* = \left\{a_\alpha : \alpha \in I_n^k\right\}$.

Definition 2 *Let $A \in \mathcal{M}_n^k$ and let X be a nonempty subset of I_n^k. The entries of A corresponding to X will be denoted by X_A. Precisely,*

$$X_A = \{a_\alpha \in A^* : \alpha \in X\}.$$

Similarly, let Y be a nonempty subset of A^*. The indices of I_n^k corresponding to Y will be denoted by $Y_{I_n^k}$. Precisely,

$$Y_{I_n^k} = \left\{\alpha \in I_n^k : a_\alpha \in Y\right\}.$$

With a slight abuse of notation, we will write $X^* = X_A$ even though X is not necessarily an element of $\mathcal{M}_n^k$. However, this should cause no confusion.

Definition 3 *Let $A \in \mathcal{M}_n^k$ and let X be a nonempty subset of I_n^k. Also, let $k' \in \mathbb{N}$ with $1 \leq k' \leq k$. Any matrix $B \in \mathcal{M}_n^{k'}$ such that $X^* = X_A$ can be placed into a one-to-one correspondence with B^* is called a matrix associated with X.*

It is clear from Definition 3 that there is a matrix B associated with X if and only if $|X| = n^{k'}$. Various subsets of A^* will play an important role. They will be used often enough to warrant some additional definitions.

Definition 4 *Let $A \in \mathcal{M}_n^k$. The main diagonal of A, denoted by $\Delta(A)$, is the set*

$$\Delta(A) = \{a_\alpha \in A^* : \alpha = c(1, 1, \ldots, 1) \text{ for some } c \in I_n\}.$$

Definition 5 *Let $A \in \mathcal{M}_n^k$. For $i \in I_k$ and $j \in I_n$, the jth face of A in the ith direction, denoted by $F_{ij}(A)$, is the set*

$$F_{ij}(A) = \{a_\alpha \in A^* : \alpha_i = j\}.$$

Depending on the situation, it is more advantageous to consider not the definition of the various subsets of A^* in terms of imposed conditions on the coordinates of the entries of A, but rather in terms of the Cartesian product of subsets of I_n. In this regard, the face $F_{ij}(A)$ can equivalently be viewed as the restriction of A to the set

$$X_{ij} = \prod_{t=1}^{k} j^{\delta_{it}} I_n^{1-\delta_{it}}.$$

In other words, $F_{ij}(A) = X_{ij}^*$. Additionally, it will be important at times to consider a particular face containing the entry a_α of a k-dimensional n-square matrix A. This leads to the following definition.

Definition 6 *Let $A \in \mathcal{M}_n^k$. For $a_\alpha \in A^*$, the ith face of A at a_α, denoted by $F_i(a_\alpha)$, is the set*

$$F_i(a_\alpha) = \{a_\beta \in A^* : \alpha_i = \beta_i\} .$$

Also, the face set of A at a_α, denoted by $\mathcal{F}_{a_\alpha}$, is the set

$$\mathcal{F}_{a_\alpha} = \{F_i(a_\alpha) : i \in I_k\} .$$

There is an intimate connection between Definition 5 and Definition 6. Precisely, the face $F_i(a_\alpha)$ can be viewed as the α_i-th face of A in the ith direction; i.e., $F_i(a_\alpha) = F_{i,\alpha_i}(A)$. This connection will be exploited often. Moreover, observe that $F_i(a_\alpha) = F_i(a_\beta)$ if and only if $\alpha_i = \beta_i$.

Remark 1 *The figures in this article may contain several highlighted vertices either to illustrate certain sets of vertices or to illustrate certain adjacencies that exist between vertices. The highlighted vertices do not necessarily indicate a proper coloring of a subset of the vertices. To avoid any confusion, it will be stated explicitly when the highlighted vertices represent a proper coloring.*

Example 1 *In Figures 1 - 3 below, the faces $F_{12}(A) = F_1(a_{231})$, $F_{23}(A) = F_2(a_{231})$, and $F_{31}(A) = F_3(a_{231})$ are illustrated for a 3-dimensional 4-square matrix A, respectively. The elements contained in the faces are highlighted in either red or blue. The element colored blue is the element at which the face is located. The light gray lines in the grid graphs are for visual purposes only and are not meant to indicate edges in a graph.*

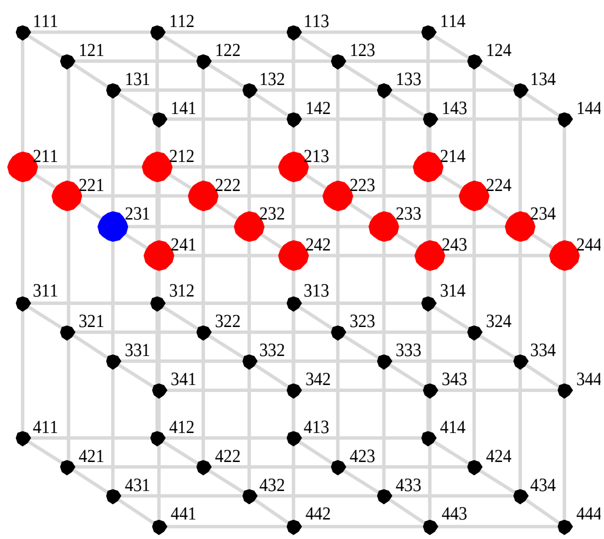

Figure 1: The face $F_{12}(A) = F_1(a_{231})$

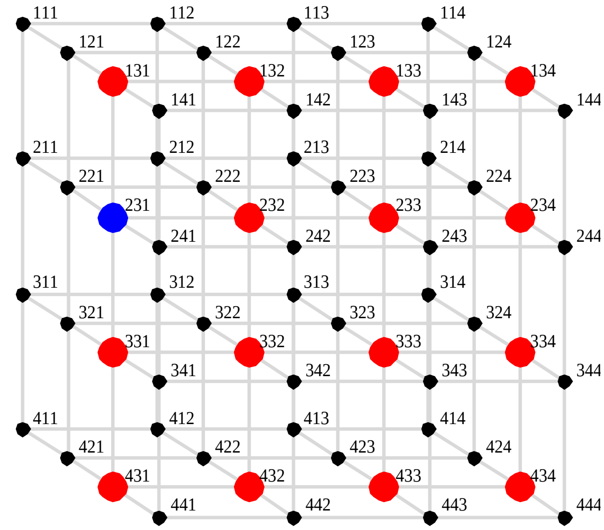

Figure 2: The face $F_{23}(A) = F_2(a_{231})$

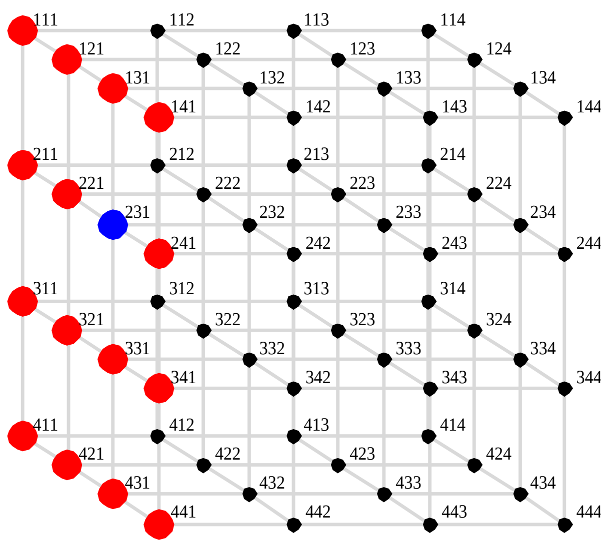

Figure 3: The face $F_{31}(A) = F_3(a_{231})$

In addition to the faces of a k-dimensional n-square matrix A, it is important to consider sets of entries that are "perpendicular" to the faces. In the same spirit as the normal line at a point is perpendicular to the tangent line at a point, the notion of a "normal" of a k-dimensional n-square matrix is defined next.

Definition 7 *Let $A \in \mathcal{M}_n^k$. Also, let $F_i = F_i(a_\alpha)$ be the ith face of A at a_α. The ith normal of A at a_α, denoted by $\eta_{F_i}^{\perp}(a_\alpha)$, is the set*

$$\eta_{F_i}^{\perp}(a_\alpha) = \left\{a_\beta \in A^* : \beta_j = \alpha_j \text{ for all } j \neq i\right\} .$$

Also, the normal set of A at a_α, denoted by $\eta^{\perp}(a_\alpha)$, is the set

$$\eta^{\perp}(a_\alpha) = \left\{\eta_{F_i}^{\perp}(a_\alpha) : i \in I_k\right\} .$$

It is not difficult to verify that the ith normal of A at a_α can be expressed in terms of the faces of A at a_α. In particular,

$$\eta_{F_i}^{\perp}(a_\alpha) = \bigcap_{t \neq i} F_t(a_\alpha) .$$

Definition 7 is clarified by the next example.

Example 2 *Figure 4 below shows the union of the normal set of a matrix $A \in \mathcal{M}_4^3$ at a_{231}. Note that the 1st normal of A at a_{231} is perpendicular to $F_1(a_{231})$, the 2nd normal of A at a_{231} is perpendicular to $F_2(a_{231})$, and the 3rd normal of A at a_{231} is perpendicular to $F_3(a_{231})$. The green and light gray lines are for visual purposes only as stated earlier. Also, the vertex at which the normal set located is indicated in blue and the remaining vertices in the normal set are highlighted in red.*

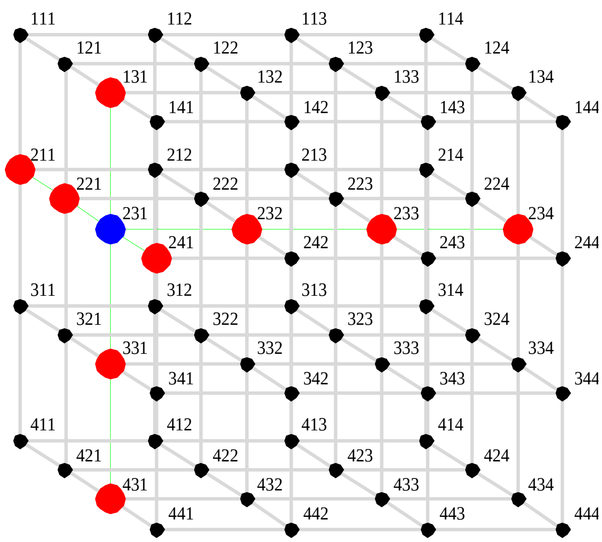

Figure 4: The normal set of A at a_{231}.

The next two definitions will prove useful in Section 3.1.

Definition 8 *Let $A \in \mathcal{M}_n^k$, where $k \geq 3$, and let $a_\alpha \in A^*$. Further let $i_1, i_2 \in I_k$, with $i_1 \neq i_2$. The (i_1, i_2)-hyperface of A at a_α, denoted by $F_{i_1}^{i_2}(a_\alpha)$, is the set*

$$\begin{aligned} F_{i_1}^{i_2}(a_\alpha) &= \left\{a_\beta \in A^* : \beta_{i_1} = \alpha_{i_1} \text{ and } \beta_{i_2} = \alpha_{i_2}\right\} \\ &= F_{i_1}(a_\alpha) \cap F_{i_2}(a_\alpha) . \end{aligned}$$

Also, the hyperface set of A at a_α, denoted by $\mathcal{F}^{a_\alpha}$, is the set

$$\mathcal{F}^{a_\alpha} = \left\{F_{i_1}^{i_2}(a_\alpha) : i_1, i_2 \in I_k \text{ and } i_1 \neq i_2\right\} .$$

In a similar fashion, the notion of a hypernormal is defined.

Definition 9 *Let $A \in \mathcal{M}_n^k$, where $k \geq 3$. Also, let $F_{i_1}(a_\alpha)$ be the i_1-st face of A at a_α. For $i_2 \in I_k$ with $i_2 \neq i_1$, the (i_1, i_2)-hypernormal of A at a_α is the set*

$$\eta^{\perp}_{F^{i_2}_{i_1}}(a_\alpha) = \left\{ a_\beta \in A^* : \beta_j = \alpha_j \text{ for all } j \neq i_1, i_2 \right\}.$$

In three dimensions, the notions of hyperface and normal are equivalent as are the notions of face and hypernormal. Additionally, the notions of faces and normals, as well as hyperfaces and hypernormals, can be expressed in terms of Cartesian products. For instance, the hypernormal $\eta^{\perp}_{F^{i_2}_{i_1}}(a_\alpha)$ can be equivalently viewed as the restriction of A to the set

$$X_{i_1, i_2}(a_\alpha) = \prod_{t=1}^{k} (\alpha_t)^{1-\delta_{i_1 t}-\delta_{i_2 t}} \, I_n^{\delta_{i_1 t}+\delta_{i_2 t}}.$$

In other words, $\eta^{\perp}_{F^{i_2}_{i_1}}(a_\alpha) = X^*_{i_1, i_2}(a_\alpha)$. Also, the hyperface $F^{i_2}_{i_1}(a_\alpha)$ can be viewed as the restriction of A to the set

$$Y_{i_1, i_2}(a_\alpha) = \prod_{t=1}^{k} (\alpha_{i_1})^{\delta_{i_1 t}} (\alpha_{i_2})^{\delta_{i_2 t}} \, I_n^{1-\delta_{i_1 t}-\delta_{i_2 t}}.$$

Thus, $F^{i_2}_{i_1}(a_\alpha) = Y^*_{i_1, i_2}(a_\alpha)$.

Definition 10 *Let $A \in \mathcal{M}_n^k$ and let $a_\alpha \in A^*$. The k-dimensional $(n-1)$-square submatrix of A determined by a_α, denoted by $A\left(|_{\mathcal{F}_{a_\alpha}}\right)$, is the restriction of A to the set*

$$I_n^k(a_\alpha) = \prod_{i=1}^{k} \left(I_n \backslash \{\alpha_i\}\right).$$

Furthermore, $A\left(|_{\mathcal{F}_{a_\alpha}}\right)$ will be written in expanded form as

$$\begin{aligned} A\left(|_{\mathcal{F}_{a_\alpha}}\right) &= A\left(F_i(a_\alpha)\,|_{i=1}^{k}\right) \\ &= A\left(F_1(a_\alpha) \mid F_2(a_\alpha) \mid \cdots \mid F_k(a_\alpha)\right). \end{aligned}$$

Definition 11 *Let $i \in I_k$. The ith projection of the set I_n^k, denoted by $\pi_i\left(I_n^k\right)$, is the set*

$$\pi_i\left(I_n^k\right) = \prod_{j=1}^{k} I_n^{\delta_{ij}}.$$

Definition 12 *Let S be any nonempty set. For $i \in I_k$, a k-dimensional n-row matrix is any function $X_i : \pi_i\left(I_n^k\right) \to S$. For $\alpha \in \pi_i\left(I_n^k\right)$, where $\alpha_j = \alpha_i^{\delta_{ij}}$ for $j \in I_k$, we write*

$$\begin{aligned} X_i(\alpha) &= x^i_\alpha \\ &= x^i_{1,1,\ldots,\alpha_i,\ldots,1,1} \end{aligned}$$

and

$$\begin{aligned} X_i &= \left[x^i_\alpha\right] \\ &= \left[x^i_{1,1,\ldots,1,\ldots,1,1},\, x^i_{1,1,\ldots,2,\ldots,1,1},\, \ldots,\, x^i_{1,1,\ldots,n,\ldots,1,1}\right], \end{aligned}$$

where the "centered" coordinate is in the ith position.

2.2 On the ordering of entries in a k-dimensional n-square matrix

Recall that the domain of a k-dimensional n-square matrix is the Cartesian product I_n^k. The set I_n^k is ordered in the usual way. For $\alpha,\, \beta \in I_n^k$, where $\alpha = (\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_k)$ and $\beta = (\beta_1,\, \beta_2,\, \ldots,\, \beta_k)$, we have $\alpha < \beta$ if and only if $\alpha_1 < \beta_1$ or there is an index $m \in I_{k-1}$ such that $\alpha_i = \beta_i$ for $i \in I_m$ and $\alpha_{m+1} < \beta_{m+1}$. Also, $\alpha = \beta$ if and only if $\alpha_i = \beta_i$ for $i \in I_k$. Therefore, the ordering of the elements of a k-dimensional n-square matrix will correspond to the natural [dictionary] ordering of its domain. Moreover, any subset a a k-dimensional n-square matrix will be ordered in the same fashion

2.3 On the cardinalities of subsets of a k-dimensional n-square matrix

In Section 2.1, various subsets of a matrix $A \in \mathcal{M}_n^k$ were defined. Cardinalities of these various subsets are determined next.

Proposition 1 *Let $A \in \mathcal{M}_n^k$. The cardinality of any normal of A is n.*

Proposition 2 *Let $A \in \mathcal{M}_n^k$ and let $a_\alpha \in A^*$. Then*

$$\left|\cup\eta^\perp(a_\alpha)\right| = k(n-1)+1.$$

Proof. Recall the normal set of A at a_α is given by

$$\eta^\perp(a_\alpha) = \left\{\eta^\perp_{F_i}(a_\alpha) : i \in I_k\right\}.$$

For $i_1,\, i_2 \in I_k$, with $i_1 \neq i_2$, observe that $\eta^\perp_{F_{i_1}}(a_\alpha) \cap \eta^\perp_{F_{i_2}}(a_\alpha) = \{a_\alpha\}$. Therefore, by Proposition 1, it follows that

$$\begin{aligned}
\left|\left[\cup\eta^\perp(a_\alpha)\right] \setminus \{a_\alpha\}\right| &= \left|\bigcup_{i=1}^{k}\left(\left[\eta^\perp_{F_i}(a_\alpha)\right]\setminus\{a_\alpha\}\right)\right| \\
&= \sum_{i=1}^{k}(n-1) \\
&= k(n-1)
\end{aligned}$$

Consequently, $\left|\cup\eta^\perp(a_\alpha)\right| = k(n-1)+1$. ■

We remark that the result of Proposition 2 can be obtained also by applying the principle of inclusion-exclusion. For,

$$\begin{aligned}
\left|\cup\eta^\perp(a_\alpha)\right| &= \left|\bigcup_{i=1}^{k}\eta^\perp_{F_i}(a_\alpha)\right| \\
&= \sum_{i\,=\,1}^{k}\left|\eta^\perp_{F_i}(a_\alpha)\right| - \sum_{i\,=\,2}^{k}(-1)^i\binom{k}{i} \\
&= nk-(k-1) \\
&= k(n-1)+1.
\end{aligned}$$

Proposition 3 *Let $A \in \mathcal{M}_n^k$. The cardinality of any face of A is n^{k-1}.*

Proposition 4 *Let $A \in \mathcal{M}_n^k$ and let $a_\alpha \in A^*$. Then*

$$|\cup \mathcal{F}_{a_\alpha}| = kn^{k-1} - \sum_{i\,=\,2}^{k} (-1)^i \binom{k}{i} n^{k-i} .$$

Proof. *Recall that the face set of A at a_α is*

$$\mathcal{F}_{a_\alpha} = \{F_i(a_\alpha) : i \in I_k\} .$$

For i_1, $i_2 \in I_k$, with $i_1 \neq i_2$, observe that

$$\begin{aligned} F_{i_1}(a_\alpha) \cap F_{i_2}(a_\alpha) &= \left\{a_\beta \in A^* : \alpha_{i_1} = \beta_{i_1} \text{ and } \alpha_{i_2} = \beta_{i_2}\right\} \\ &= F_{i_1}^{i_2}(a_\alpha), \end{aligned}$$

the (i_1, i_2)-hyperface of A at a_α. Therefore,

$$|F_{i_1}(a_\alpha) \cap F_{i_2}(a_\alpha)| = n^{k-2}.$$

More generally, for i_{s_1}, $i_{s_2} \in I_k$, with $i_{s_1} \neq i_{s_2}$, observe that

$$\bigcap_{s=1}^{t} F_{i_s}(a_\alpha) = n^{k-t}.$$

Hence, by applying Proposition 3 and the principle of inclusion-exclusion, we find that

$$\begin{aligned} |\cup \mathcal{F}_{a_\alpha}| &= \left| \bigcup_{i\,=\,1}^{k} F_i(a_\alpha) \right| \\ &= \sum_{i=1}^{k} |F_i(a_\alpha)| - \sum_{i=2}^{k} (-1)^i \binom{k}{i} n^{k-i} \\ &= kn^{k-1} - \sum_{i=2}^{k} (-1)^i \binom{k}{i} n^{k-i} . \end{aligned}$$

This completes the proof. ■

3 Matrix graphs

In the previous section, the notion of a k-dimensional n-square matrix has been introduced and some of the basic properties of these matrices have be investigated. In the current section, the notion of a k-dimensional n-square matrix is used to define a particular graph construction. A graph constructed in such a fashion is called a matrix graph. There are two primary ways to construct a graph from a given k-dimensional n-square matrix. For lack of better terminology, they will be referred to as a Type I matrix graph and a Type II matrix graph. Each of these constructions will be similar in nature; but, one construction, namely a Type I matrix graph, will prove to be more useful for our purposes. Therefore, more attention will be devoted to Type I matrix graphs.

3.1 Construction of Type I and Type II matrix graphs

The construction of Type I and Type II matrix graphs depend on the selection of a k-dimensional n-square matrix. Recall that we have imposed the condition that $n \geq k$.

Type I Matrix Graph Let $A = [a_\alpha] \in \mathcal{M}_n^k$. Denote a Type I matrix graph by $G_1(A)$. Define the vertex set of $G_1(A)$ by setting $V(G_1(A)) = A^*$ and by defining the edge set of $G_1(A)$ according to the following rule:

$$a_\alpha a_\beta \in E(G_1(A)) \text{ if and only if } a_\beta \notin \cup \mathcal{F}_{a_\alpha}.$$

The graph $G_1(A)$ will be called a Type I matrix graph determined by A. We remark in a Type I matrix graph for $A \in \mathcal{M}_n^2$, that $a_{ij}a_{rs} \in E(G_1(A))$ if and only if $r \neq i$ and $s \neq j$. In other words, vertex a_{ij} is adjacent to all vertices that remain when the ith row and jth column are deleted from the $n \times n$ matrix A. The definition of the edge set of a Type I matrix graph is the generalization of this notion. Figure 5 below shows all vertices (in red) adjacent to vertex a_{221} (in blue) in a Type I matrix graph for a matrix $A \in \mathcal{M}_3^3$. The green lines demonstrate the adjacencies and the light gray lines are for visual purposes only. Note that some edges (in green) may overlap. Recall as in Remark 1, we are not indicating a coloration of the vertices.

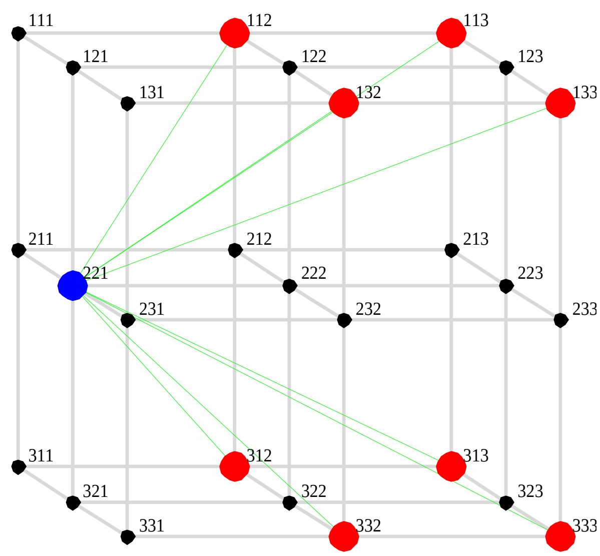

Figure 5: Vertices adjacent to vertex a_{221}.

The next two illustrations represent the entire Type I matrix graph for $A \in \mathcal{M}_3^3$. The visualizations in Figure 6 and Figure 7 below are the initially conceived matrix form and a circular embedding, respectively. Also illustrated in these two figures is a 3-coloring of $G_1(A)$.

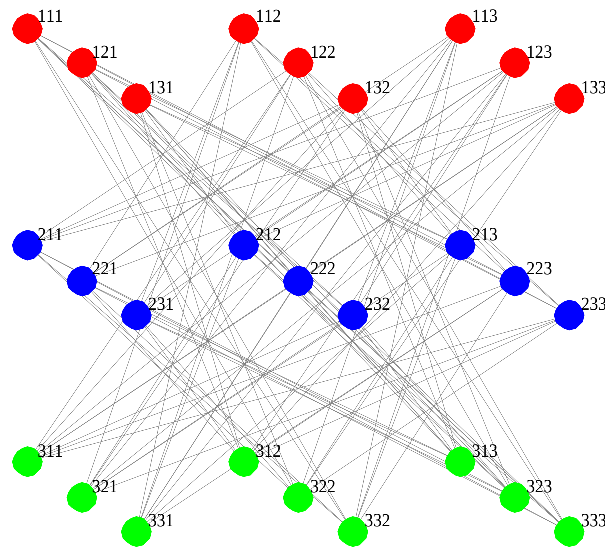

Figure 6: $G_1(A)$ as originally conceived.

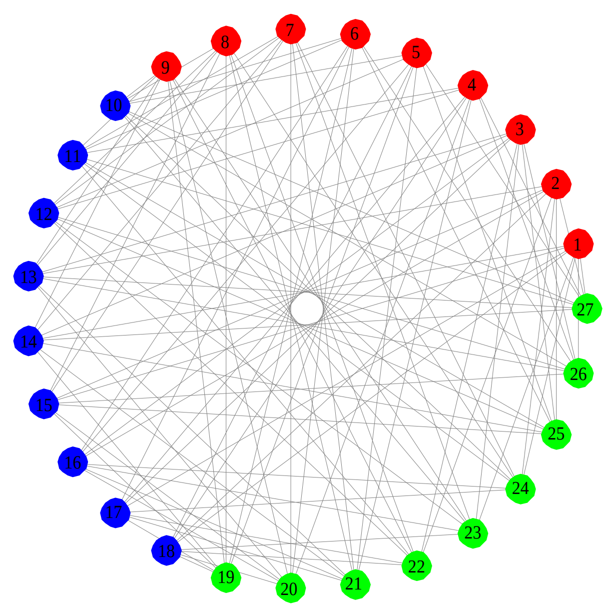

Figure 7: Circular embedding of $G_1(A)$.

Type II Matrix Graph Let $A = [a_\alpha] \in \mathcal{M}_n^k$. Denote a Type II matrix graph by $G_2(A)$. Define the vertex set of $G_2(A)$ by setting $V(G_2(A)) = A^*$ and by defining the edge set of $G_2(A)$ by the rule

$$a_\alpha a_\beta \in E(G_2(A)) \text{ if and only if } a_\beta \notin \cup\eta^\perp(a_\alpha).$$

The graph $G_2(A)$ will be called a Type II matrix graph determined by A. Figure 8 below shows all vertices (in red) adjacent to vertex a_{231} (in blue) in a Type II matrix graph for a matrix $A \in \mathcal{M}_3^3$. The green lines demonstrate the adjacencies and the light gray lines are for visual purposes only. Note that some edges (in green) may overlap. Recall as in Remark 1, we are not indicating a coloration of the vertices.

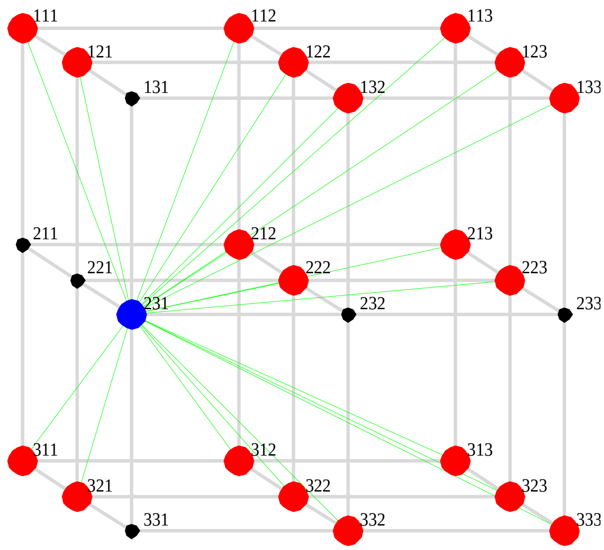

Figure 8: Vertices adjacent to vertex a_{231}.

The next two illustrations represent the entire Type II matrix graph for a matrix $A \in \mathcal{M}_3^3$. The visualizations in Figure 9 and Figure 10 below are the initially conceived matrix form and a circular embedding, respectively. Also illustrated in these two figures is a 9-coloring of $G_2(A)$.

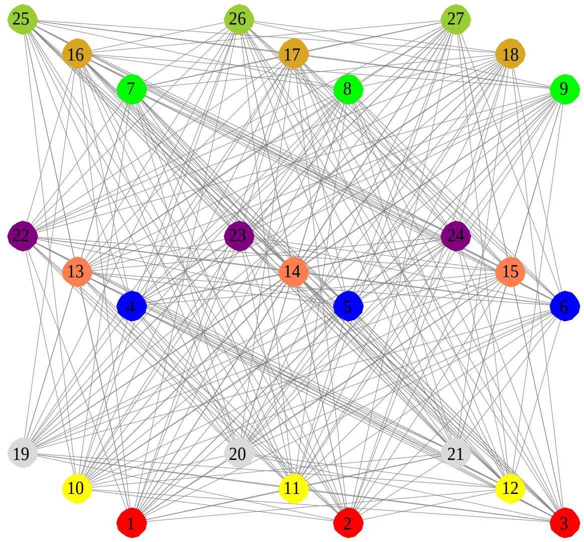

Figure 9: $G_2(A)$ as originally conceived.

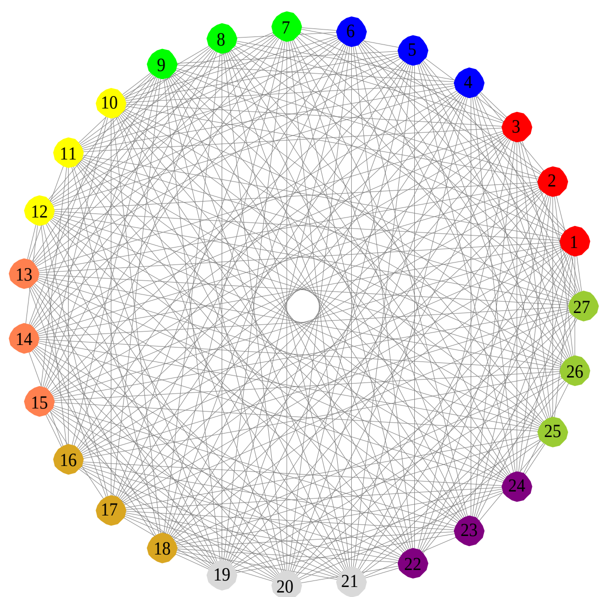

Figure 10: Circular embedding of $G_2(A)$.

3.2 On subgraphs of a Type I matrix graph

Consider $A \in \mathcal{M}_n^{k+1}$, where $k \geq 2$ and $n \geq k+1$. Let $G_1(A)$ be a Type I matrix graph. We would like to establish the existence of a subset X^* of A^* such that the induced subgraph $G_1(A)[X^*]$ is isomorphic to a Type I matrix graph determined by a matrix $C \in \mathcal{M}_n^k$. In this case, C will be an associated matrix for $X^*_{I_n^k}$, the indices of I_n^{k+1} corresponding to X^*. For convenience, this Type I matrix graph will be denoted by $H_1(C)$ so that $G_1(A)[X^*] \cong H_1(C)$. In fact, it will be shown that $V(G_1(A))$ can be partitioned into n subsets, each one of which determines an induced subgraph of $G_1(A)$ that is isomorphic to $H_1(C)$.

Proposition 5 *Let $k \in \mathbb{N}$, where $k \geq 2$, and let $A \in \mathcal{M}_n^{k+1}$. The Type I matrix graph $G_1(A)$ contains an induced subgraph isomorphic to a Type I matrix graph $H_1(C)$ determined by a matrix $C \in \mathcal{M}_n^k$.*

Proof. Let $k \in \mathbb{N}$ and let $A \in \mathcal{M}_n^{k+1}$, where $k \geq 2$. Consider a Type I matrix graph $G_1(A)$. Select an arbitrary permutation of I_n, say $\lambda = \lambda_1\lambda_2\cdots\lambda_n$, and also select an arbitrary 2-permutation of I_k, say $\rho = \rho_1\rho_2$. The set

$$T(\lambda;\, \rho) = \left\{ \prod_{i=1}^{k+1} j^{\delta_{i\rho_1}} \lambda_j^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}} : j \in I_n \right\}$$

consists of n subsets of I_n^{k+1} each one of which corresponds to a hyperface of A, namely $F_{\rho_1}^{\rho_2}(a_\alpha)$ for any a_α satisfying $\alpha_{\rho_1} = j$ and $\alpha_{\rho_2} = \lambda_j$. Since two of the $k+1$ coordinate positions, namely ρ_1 and ρ_2, of each member of $T(\lambda;\, \rho)$ are fixed, it is clear that each member of $T(\lambda;\, \rho)$ is a nonempty subset of I_n^{k+1} having cardinality n^{k-1}. Define $X = \cup T(\lambda;\, \rho)$. Then $|X| = n \cdot n^{k-1} = n^k$ and; moreover, $n \geq k+1 > k$. Hence, there exists a matrix $C \in \mathcal{M}_n^k$ associated with X. We claim that $G_1(A)[X^*] \cong H_1(C)$, where $H_1(C)$ is a Type I matrix graph determined by C. Let $\mathcal{F}_{a_\alpha}^{X^*}$ be the face set of a_α in X^*. The claim follows immediately from the fact that for all $a_\alpha,\, a_\beta \in X^*$, we have $a_\beta \in \mathcal{F}_{a_\alpha}^{X^*}$ if and only if $a_\beta \in \mathcal{F}_{a_\alpha}$. ■

Corollary 1 *Let $k \in \mathbb{N}$, where $k \geq 2$, and let $G_1(A)$ be a Type I matrix graph, where $A \in \mathcal{M}_n^{k+1}$. There exists a Type I matrix graph $H_1(C)$, for some matrix $C \in \mathcal{M}_n^k$, and a partition $\mathcal{P}$ of $V(G_1(A))$ such that $G_1(A)[P] \cong H_1(C)$ for each $P \in \mathcal{P}$.*
Proof. *We have demonstrated the existence of a subset of A^* which admits an induced subgraph of $G_1(A)$ that is isomorphic to a Type I matrix graph having dimension one less than that of $G_1(A)$. It is now shown that $V(G_1(A))$ can be partitioned in such a way that each element of the partition admits an induced subgraph of $G_1(A)$ that is isomorphic to a Type I matrix graph having dimension one less than that of $G_1(A)$. Select an arbitrary Latin square having order n, say*

$$L_n = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_n \end{bmatrix},$$

where $\Lambda_i = (\lambda_{i1},\, \lambda_{i2},\, \ldots,\, \lambda_{in})$ is the ith row of the Latin square L_n. For a fixed 2-permutation of I_k, say $\rho = \rho_1\rho_2$, define

$$T(L_n;\, \rho) = \bigcup_{q=1}^{n} \{T(\Lambda_q;\, \rho)\},$$

where, as in the proof Proposition 5 above, the set $T(\Lambda_q;\, \rho)$ is defined as

$$T(\Lambda_q;\, \rho) = \left\{ \prod_{i=1}^{k+1} j^{\delta_{i\rho_1}} \lambda_{qj}^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}} : j \in I_n \right\}.$$

To see that the set $T(L_n;\rho)$ is a pairwise disjoint collection of subsets of I_n^k, suppose to the contrary that $\alpha \in T(\Lambda_{q_1};\rho) \cap T(\Lambda_{q_2};\rho)$, where $q_1, q_2 \in I_n$ with $q_1 \neq q_2$. By the definition of $T(\Lambda_q;\rho)$, the $(k+1)$-tuple α would be expressible in two ways as

$$\alpha = j_1^{\delta_{i\rho_1}} \lambda_{q_1 j_1}^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}}$$

and

$$\alpha = j_2^{\delta_{i\rho_1}} \lambda_{q_2 j_2}^{\delta_{i\rho_2}} I_n^{1-\delta_{i\rho_1}-\delta_{i\rho_2}},$$

for some $j_1, j_2 \in I_n$. Necessarily, $j_1 = j_2$ and $\lambda_{q_1 j_1} = \lambda_{q_2 j_2}$. But since L_n is a Latin square, it would have to be that $q_1 = q_2$ contrary to the assumption that $q_1 \neq q_2$. This is a contradiction and confirms that $T(L_n;\rho)$ is a pairwise disjoint collection of nonempty subsets of I_n^k. Moreover,

$$|\cup T(L_n;\rho)| = n \cdot n^k = n^{k+1}$$

so that $\cup T(L_n;\rho) = I_n^{k+1}$. Finally, for $q \in I_n$, define $X_q = \cup T(\Lambda_q;\rho)$. By Proposition 5, we find that $G_1(A)\left[X_q^*\right] \cong H_1(C)$, for some $C \in \mathcal{M}_n^k$. It follows that the collection

$$\mathcal{P} = \left\{X_q^* : q \in I_n\right\}$$

is such a partition. ■

3.3 On independent subsets in a Type I matrix graph

First, a few general observations are noted regarding the adjacencies and nonadjacencies in a Type I matrix graph $G_1(A)$. Recall that the face set of A at a_a is given by $\mathcal{F}_{a_\alpha} = \{F_i(a_\alpha) : i \in I_k\}$. Suppose that $F_{i_0}(a_\alpha) \in \mathcal{F}_{a_\alpha}$ and $a_\gamma \notin F_{i_0}(a_\alpha)$. Then there is an element a_β of A^* such that $a_\beta \in F_{i_0}(a_\alpha)$ and $a_\gamma \in \eta_{F_{i_0}}^{\perp}(a_\beta)$. Therefore, in $F_{i_0}(a_\alpha)$, the vertex a_γ fails to be adjacent to each vertex in the set

$$\bigcup_{j \,\neq\, i_0}^{k} \eta_{F_j}^{\perp}(a_\beta).$$

By Proposition 1, note that

$$\left|\bigcup_{j \,\neq\, i_0}^{k} \eta_{F_j}^{\perp}(a_\beta)\right| = (k-1)(n-1)+1.$$

Moreover, a_γ is adjacent to all other vertices in the face $F_{i_0}(a_\alpha)$. Consequently, a_γ satisfies the following:

(*i*) a_γ is adjacent to exactly $n^{k-1} - [(k-1)(n-1)+1]$ vertices in $F_{i_0}(a_\alpha)$.
(*ii*) a_γ is not adjacent to exactly $(k-1)(n-1)+1$ vertices in $F_{i_0}(a_\alpha)$.

Let $A \in \mathcal{M}_n^k$ and consider a Type I matrix graph $G_1(A)$. We now consider certain independent subsets of $V(G_1(A))$. The next proposition is straightforward.

Proposition 6 *The faces of a Type I matrix graph are independent.*
Proof. *Let $A \in \mathcal{M}_n^k$ and let $G_1(A)$ be a Type I matrix graph. Suppose that $a_\alpha \in A^*$ and consider an arbitrary face containing a_α, say $F_{i_0}(a_\alpha)$ for some $i \in I_k$. If $a_\beta \in A^*$ and $a_\beta \in F_{i_0}(a_\alpha)$, then $a_\beta \in \cup\mathcal{F}_{a_\alpha}$. Therefore, $a_\alpha a_\beta \notin E(G_1(A))$, which follows immediately from the definition of the edge set of $G_1(A)$:*

$$a_\alpha a_\beta \in E(G_1(A)) \text{ if and only if } a_\beta \notin \cup\mathcal{F}_{a_\alpha}.$$

As a result, the faces of a Type I matrix graph are independent subsets of $V(G_1(A))$. ■

Corollary 2 *The normals of a Type I matrix graph are independent.*

Proposition 7 *Let $G_1(A)$ be a Type I matrix graph, where $A \in \mathcal{M}_n^2$. Then $\alpha(G_1(A)) = n$.*
Proof. *Note that in two dimensions, the notions of faces and normal are equivalent. Now, let I be an arbitrary independent subset of $V(G_1(A))$. Since $n \geq 2$, we can assume that $|I| \geq 2$. Moreover, by Proposition 6, each face of A is an independent subset of $V(G_1(A))$. It suffices to prove $I \subseteq F$ for some face F. To this end, suppose that a_α and a_β are distinct vertices in I with $\alpha = (i_1, j_1)$ and $\beta = (i_2, j_2)$. Observe that either $i_1 = i_2$ or $j_1 = j_2$, but not both. This is because otherwise, $a_\alpha a_\beta \in E(G_1(A))$. Without loss of generality, we may assume that $\{a_{ij_1}, a_{ij_2}\} \subseteq I$ and that $j_1 \neq j_2$. Now consider an arbitrary $a_{kj} \in I$ and suppose to the contrary that $k \neq i$. Now, either $j \neq j_1$ or $j \neq j_2$. Consequently, either $a_{kj}a_{ij_1} \in E(G_1(A))$ or $a_{kj}a_{ij_2} \in E(G_1(A))$, which contradicts that fact that $a_{kj} \in I$. It follows that $I \subseteq F_i(a_\alpha) = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$. Therefore, $|I| \leq n$ and $\alpha(G_1(A)) = n$.* ■

Although there will be no direct appeal to the next proposition, its proof might shed some light on a generalization of Theorem 4 in Section 4 below.

Proposition 8 *Let $G_1(A)$ be a Type I matrix graph, where $A \in \mathcal{M}_n^3$. Then $\alpha(G_1(A)) = n^2$.*
Proof. *Let I be an arbitrary independent subset of $V(G_1(A))$. Since $n \geq 3$, we can assume that $|I| \geq 2$. Suppose now that $a_\alpha, a_\beta \in I$. Because $a_\beta \in \cup\mathcal{F}_{a_\alpha}$, we can further assume, without loss of generality, that $a_\beta \in F_{i_1}(a_\alpha)$, for some direction $i_1 \in I_3$. First, the characteristics of a vertex $a_\gamma \in I \backslash F_{i_1}(a_\alpha)$ are established. There are two cases to consider.*

Case 1 *The vertices a_α and a_β are contained in the same normal.*
In this case, it will be demonstrated that a_α, a_β, and a_γ are all contained in some face of A. There exists a direction $i_2 \in I_3 \backslash \{i_1\}$ such that

$$a_\beta \in F_{i_1}(a_\alpha) \cap \eta^{\perp}_{F_{i_2}}(a_\alpha). \tag{1}$$

We assert that there must exist a face F of A such that

$$a_\gamma \in F \in \mathcal{F}_{a_\alpha} \cap \mathcal{F}_{a_\beta}.$$

In fact, it will be shown that $F = F_{i_3}(a_\alpha)$. To see this, observe that because $k = 3$, we have

$$\eta^{\perp}_{F_{i_2}}(a_\alpha) = F_{i_1}(a_\alpha) \cap F_{i_3}(a_\alpha),$$

where i_3 is the only remaining direction; i.e., $\{i_3\} = I_3 \backslash \{i_1, i_2\}$. Next, observe from (1) above, the following three conditions hold:

$$\begin{array}{rl} (i) & F_{i_1}(a_\beta) = F_{i_1}(a_\alpha) \\ (ii) & F_{i_3}(a_\beta) = F_{i_3}(a_\alpha) \\ (iii) & \eta^{\perp}_{F_{i_2}}(a_\beta) = \eta^{\perp}_{F_{i_2}}(a_\alpha). \end{array} \tag{2}$$

From (1) again, it is emphasized that $F_{i_2}(a_\alpha) \neq F_{i_2}(a_\beta)$. Because $k = 3$, the face sets at a_α and at a_β are given by

$$\mathcal{F}_{a_\alpha} = \{F_{i_1}(a_\alpha), F_{i_2}(a_\alpha), F_{i_3}(a_\alpha)\}$$

and

$$\mathcal{F}_{a_\beta} = \{F_{i_1}(a_\beta), F_{i_2}(a_\beta), F_{i_3}(a_\beta)\}.$$

Hence, from (i) and (ii) in (2) above, it follows that

$$\mathcal{F}_{a_\alpha} \cap \mathcal{F}_{a_\beta} = \{F_{i_1}(a_\alpha), F_{i_3}(a_\alpha)\}.$$

Continuing, we find that $a_\gamma \in F_{i_3}(a_\alpha) = F_{i_3}(a_\beta)$ because $a_\gamma \in F \in \mathcal{F}_{a_\alpha} \cap \mathcal{F}_{a_\beta}$ and we have assumed that $a_\gamma \notin F_{i_1}(a_\alpha)$. Therefore, it follows that a_α, a_β, and a_γ are all contained in a single face of A. □

Case 2 *The vertices a_α and a_β are not contained in the same normal.*

In this case, it will be demonstrated that a_α, a_β, and a_γ are all contained in the union of a normal set of A. Note that we are still under the assumption that $a_\beta \in F_{i_1}(a_\alpha)$. From this assumption, we find that

$$\begin{array}{rl} (iv) & \beta_{i_1} = \alpha_{i_1} \text{ and } \beta_i \neq \alpha_i \text{ for } i \neq i_1, \text{ and} \\ (v) & a_\alpha,\, a_\beta \in F_{i_1}(a_\alpha) = F_{i_1, \alpha_{i_1}}(A) = F_{i_1, \beta_{i_1}}(A)\,. \end{array} \tag{3}$$

Now, define the set

$$N = \left[\cup \eta^\perp(a_\alpha)\right] \cap \left[\cup \eta^\perp(a_\beta)\right].$$

We remark the elements of N have coordinates that agree with the coordinates of a_α in all but one position, and agree with the coordinates of a_β in all but one position, but the positions (one position from a_α and one position from a_β) of disagreement are not necessarily the same one position. Next we establish the existence of a vertex a_μ such that

$$\begin{array}{rl} (vi) & a_\mu \in N \\ (vii) & a_\gamma \in \eta^\perp_{F_{i_1}}(a_\mu)\,. \end{array} \tag{4}$$

Should both conditions of (4) *hold, it would then be the case that a_α, a_β, and a_γ are all contained in the union of a normal set of A, namely $\cup\eta^\perp(a_\mu)$. Since any two vertices in the union of a normal set are contained in at least one face of A, it is clear that the union of every normal set is an independent subset of $V(G_1(A))$. To prove the existence of such a vertex a_μ, we proceed as follows. Set $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$. Note that $I_3 = \{1, 2, 3\} = \{i_1, i_2, i_3\}$ but it may not be the case that $i_t = t$ for $t = 1, 2, 3$. Since $k = 3$ and $|N| = 2$, there are exactly two possibilities for a_μ. They are determined by defining the coordinates of a_μ. The first is given by*

$$\begin{array}{l} \mu_{i_1} = \alpha_{i_1} = \beta_{i_1} \\ \mu_{i_2} = \alpha_{i_2} \\ \mu_{i_3} = \beta_{i_3}, \end{array} \tag{5}$$

and the second is given by

$$\begin{array}{l} \mu_{i_1} = \beta_{i_1} = \alpha_{i_1} \\ \mu_{i_2} = \beta_{i_2} \\ \mu_{i_3} = \alpha_{i_3}. \end{array} \tag{6}$$

There is a permutation σ of I_3 such that $\mu = \left(\mu_{\sigma(i_1)}, \mu_{\sigma(i_2)}, \mu_{\sigma(i_3)}\right)$. From either (5) *or* (6), *it follows immediately that $a_\mu \in F_{i_1}(a_\alpha)$. Moreover, observe that μ agrees with α in all but one position and that μ agrees with β in all but one position. Hence, we find that $a_\mu \in \left[\cup\eta^\perp(a_\alpha)\right] \cap \left[\cup\eta^\perp(a_\beta)\right]$; i.e., $a_\mu \in N$. Thus, (vi) holds in* (4).

Next, it is demonstrated that $a_\gamma \in \eta^\perp_{F_{i_1}}(a_\mu)$. It suffices to show that γ agrees with μ in all but one position. In fact, γ does not agree with μ in position i_1 and γ agrees with μ in positions i_2 and i_3. To see this, observe that in either case for a_μ, we have $a_\mu \in F_{i_1}(a_\alpha)$ so that $\mu_{i_1} = \alpha_{i_1}$. However, $a_\gamma \in I \backslash F_{i_1}(a_\alpha)$ so that $\gamma_{i_1} \neq \alpha_{i_1}$. This confirms that γ does not agree with μ in position i_1. Next, it is shown that γ agrees with μ in positions i_2 and i_3. This requires more effort. Since $a_\gamma \in I$, there are faces F and F' with

$$a_\gamma \in F \in \mathcal{F}_{a_\alpha} = \{F_{i_1}(a_\alpha), F_{i_2}(a_\alpha), F_{i_3}(a_\alpha)\}$$

and

$$a_\gamma \in F' \in \mathcal{F}_{a_\beta} = \{F_{i_1}(a_\beta), F_{i_2}(a_\beta), F_{i_3}(a_\beta)\}.$$

Recall that we also have $a_\gamma \in I \backslash F_{i_1}(a_\alpha) = I \backslash F_{i_1}(a_\beta)$*. We assert that* a_γ *is a member of both the symmetric difference* $F_{i_2}(a_\alpha) \otimes F_{i_3}(a_\alpha)$ *and the symmetric difference* $F_{i_2}(a_\beta) \otimes F_{i_3}(a_\beta)$*. To prove the assertion, we first suppose to the contrary that* $a_\gamma \in F_{i_2}(a_\alpha) \cap F_{i_3}(a_\alpha)$*. Then* $\gamma_{i_2} = \alpha_{i_2}$ *and* $\gamma_{i_3} = \alpha_{i_3}$ *so that* $a_\gamma \in \eta^{\perp}_{F_{i_1}}(a_\alpha)$*. Since* $\gamma_{i_1} \neq \alpha_{i_1}$*, it follows that* $a_\beta a_\gamma \in E(G_1(A))$ *contrary to* I *being an independent subset of* $V(G_1(A))$*. For,* γ *agrees with* α *in all positions except position* i_1 *and consequently in no position does* γ *agree with* β*, which follows from (iv) in* (3) *above. Therefore,* $a_\gamma \notin F_{i_2}(a_\alpha) \cap F_{i_3}(a_\alpha)$ *and similarly* $a_\gamma \notin F_{i_2}(a_\beta) \cap F_{i_3}(a_\beta)$*. The assertion is now proven. Next, we suppose that* $a_\gamma \in F_{i_2}(a_\alpha)$*. Then* $\gamma_{i_2} = \alpha_{i_2}$*. From this, it cannot be that* $a_\gamma \in F_{i_2}(a_\beta)$ *because this would imply that* $\gamma_{i_2} = \beta_{i_2}$ *and hence* $\alpha_{i_2} = \beta_{i_2}$*, which is a contradiction of (iv) in* (3) *above. Thus, it must be the case that* $a_\gamma \in F_{i_3}(a_\beta)$ *so that* $\gamma_{i_3} = \beta_{i_3}$*. It follows that*

$$\begin{array}{rl} (viii) & \gamma_{i_1} \neq \alpha_{i_1} \text{ and } \gamma_{i_1} \neq \beta_{i_1} \\ (ix) & \gamma_{i_2} = \alpha_{i_2} \\ (x) & \gamma_{i_3} = \beta_{i_3} \end{array} . \tag{7}$$

The conditions in (5) *and* (6) *the three conditions in* (7) *imply that* $a_\gamma \in \eta^{\perp}_{F_{i_1}}(a_\mu)$ *so that (vii) holds in* (4) *above. Therefore,* $a_\alpha,\, a_\beta,\, a_\gamma \in \cup \eta^{\perp}(a_\mu)$*.* □

Now, since $n \geq k = 3$*, it follows that*

$$\begin{aligned} k(n-1)+1 &\leq n(n-1)+1 \\ &= n^2 - n + 1 \\ &< n^2. \end{aligned}$$

Moreover, $k(n-1)+1$ *is precisely the number of vertices contained in the union of a normal set by Proposition 2 . From this fact, it can assumed that if* I *is an independent set that contains more than* $k(n-1)+1$ *vertices, then there exist two vertices of* I *that are not contained in the same normal. Hence, if the vertices of* I *are not all in the same face, it would follow by Case II that the independent set* I *would be contained in the union of a normal set contradicting the fact that* I *has more than* $k(n-1)+1$ *vertices. Therefore,* I *is contained in some face so that* $|I| \leq n^2$*. We conclude that* $\alpha(G_1(A)) = n^2$ *since the faces of a Type I matrix graph are independent subsets having* n^2 *vertices.* ■

Corollary 3 *Let* $k \leq 3$*. The faces of a Type I matrix graph determine independent subsets of maximum cardinality.*

Theorem 1 *Let* $G_1(A)$ *be a Type I matrix graph, where* $A \in \mathcal{M}^k_n$*. Then* $\alpha(G_1(A)) = n^{k-1}$*.*
Proof. *The proof proceeds by induction on* k*. By Proposition 7 and Proposition 8, the result holds for* $k = 2$ *and* $k = 3$*. Inductively assume the result holds for* $k = p$*, where* $p \geq 3$*. It is shown that the result holds for* $k = p + 1$*. Let* $A \in \mathcal{M}^{k+1}_n$*, where* $n \geq k + 1$ *and consider a Type I matrix graph* $G_1(A)$ *and an arbitrary independent subset* I *of* $V(G_1(A))$*. It must be shown that* $|I| \leq n^k$*. By Corollary 1,* $V(G_1(A))$ *can be partitioned into* n *subsets, each subset of which determines an induced subgraph of* $G_1(A)$ *that is isomorphic to a Type I matrix graph* $H_1(C)$*, where* $C \in \mathcal{M}^k_n$*. Call such a partition* $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ *so that* $G_1(A)[P_i] \cong H_1(C)$ *for each* $i \in I_n$*. Observe that*

$$I = \bigcup_{i=1}^{n} (I \cap P_i).$$

By the inductive hypothesis, $\alpha(H_1(C)) = n^{k-1}$ so that $|I \cap P_i| \leq n^{k-1}$ for each $i \in I_n$. Therefore,

$$|I| = \sum_{i=1}^{n} |I \cap P_i| \leq \sum_{i=1}^{n} n^{k-1} = n^k.$$

Because the faces of $G_1(A)$ are independent subsets having n^k elements, it follows that

$$\alpha(G_1(A)) = n^k.$$

This completes the proof. ■

Corollary 4 *The faces of a Type I matrix graph determine independent subsets of maximum cardinality.*

We are now in the position to determine the chromatic number of a Type I matrix graph.

Lemma 1 *Let $G_1(A)$ be a Type I matrix graph, where $A \in \mathcal{M}_n^k$. Then $G_1(A)[\Delta] \cong K_n$, where Δ is the main diagonal of A.*
Proof. *Let $a_\alpha, a_\beta \in \Delta$. Then $\alpha = c(1, 1, \ldots, 1)$ and $\beta = d(1, 1, \ldots, 1)$ for some $c, d \in I_n$ with $c \neq d$. Therefore, it is immediately clear that*

$$a_\beta \notin \cup \mathcal{F}_{a_\alpha}$$

since there does not exist a position in which α and β agree. Because $|\Delta| = n$, it follows that the subgraph of $G_1(A)$ induced by Δ is isomorphic to K_n. ■

Theorem 2 *Let $A \in \mathcal{M}_n^k$. The Type I matrix graph $G_1(A)$ is n-chromatic.*

Proof. By Lemma 1, $\chi(G_1(A)) \geq n$. Now, the faces of A are independent subsets of A^*. Therefore, an n-coloring of $G_1(A)$ can be exhibited by coloring each face $F_{i_0 j}$ with color c_j for $j = 1, 2, \ldots, n$, where $i_0 \in I_k$ is a fixed direction. It follows that $\chi(G_1(A)) = n$. ■

Remark 2 *Theorem 2 can also be obtained as a result of Proposition 1. For,*

$$\chi(G_1(A)) \geq \frac{|G_1(A)|}{\alpha(G_1(A))} = \frac{n^k}{n^{k-1}} = n$$

and by coloring the faces as in the proof of Theorem 2, the desired result is obtained.

Proposition 9 *Let $X = \{a_{\alpha^1}, a_{\alpha^2}, \ldots, a_{\alpha^t}\}$ be a set of t vertices in a Type I matrix graph $G_1(A)$, where $A \in \mathcal{M}_n^k$ and $1 \leq t \leq n$. For $i = 1, 2, \ldots, k$, define $Z_i = \left\{\alpha_i^j : j \in I_t\right\}$. If $|Z_i| = t$ for all $i \in I_k$, then $G_1(A)[Z_i] \cong K_t$.*
Proof. *The condition $|Z_i| = t$ for all $i \in I_k$ implies that there do not exist indices $j_1, j_2 \in I_t$ and an index $i_0 \in I_k$ for which $\alpha_{i_0}^{j_1} = \alpha_{i_0}^{j_2}$. Hence,*

$$a_{\alpha^{j_1}} \notin \cup \mathcal{F}_{a_{\alpha^{j_2}}}.$$

Therefore, $a_{\alpha^{j_1}} a_{\alpha^{j_2}} \in E(G_1(A))$ and it follows that $G_1(A)[Z_i] \cong K_t$. ■

4 On a generalization of completely independent critical cliques

The notation and terminology contained in this section are not standard. Therefore, some additional definitions are required. Recall that a vertex $v \in V(G)$ is a critical vertex of G provided that the chromatic number of G decreases upon the removal of v. In fact, the chromatic number decreases be exactly one whenever v is a critical vertex; i.e., $\chi(G-v) = \chi(G) - 1$. Observe that the induced subgraph $G[\{v\}]$ satisfies $G[\{v\}] \cong K_1$. There is a natural generalization of this concept.

Definition 13 *Let K be an r-clique of G. Then K is a critical r-clique of G, written K_r^c, provided that $\chi(G-K) = \chi(G) - r$.*

It is straightforward to prove any subgraph K of order r that satisfies the equation $\chi(G-K) = \chi(G) - r$ is necessarily isomorphic to K_r. Recall from Section 1 that a set U of vertices is independent provided that no two vertices in U are adjacent. Equivalently, U is an independent subset of $V(G)$ whenever the induced subgraph $G[U]$ is isomorphic to an empty graph. The definition below of completely independent critical cliques can be viewed as a generalization of an independent set of vertices provided we consider each vertex in an independent set U as an induced subgraph of G isomorphic to K_1. However, we shall not adopt this point of view.

Definition 14 *Let K_r^c and K_s^c be two critical cliques of order r and s, respectively. Then K_r^c and K_s^c are completely independent provided $N(v) \cap V(K_s^c) = \varnothing$ for every vertex $v \in V(K_r^c)$.*

In Definition 14, K_r^c and K_s^c are completely independent provided that $\chi(G-K_r^c) = \chi(G) - r$ and $\chi(G-K_s^c) = \chi(G) - s$, and, moreover, no vertex in K_r^c is adjacent to a vertex in K_s^c. The motivation for this definition arises out of its connection with a conjecture of Lovász in [6] that the only vertex double-critical graph is the complete graph. The double-critical conjecture has been proven in the affirmative by Stiebitz in [14] only in the case of a 5-chromatic double-critical graph. A more general statement that includes the conjecture of Lovász as a special case is the Erdős-Lovász Tihany conjecture. This more general conjecture and a brief history of some of the known results can be found in [8]. Related results for quasi-line graphs are given in [1]. The edge analogue of this conjecture has been resolved in the affirmative and can be found in [9] and [13]. It seems reasonable to believe that for a single critical clique K_r^c there would be many edges from K_r^c to $G - K_r^c$. Thus it might seem just as reasonable to believe for a family of critical cliques, that there would be many edges from one critical clique to the other. The results of this paper declare this is not the case. If $\mathcal{K} = \left\{K_{r_\alpha}^c : \alpha \in \Lambda\right\}$ is an indexed family of critical r_α-cliques, then $\mathcal{K}$ is said to be a family of completely independent critical cliques provided that the elements of $\mathcal{K}$ are pairwise completely independent critical cliques. In [12], the existence of $\mathcal{K}$ for the case $|\mathcal{K}| = 2$ was addressed and it was demonstrated that there exists a vertex k-critical graph admitting two completely independent critical cliques having orders r and s for any r and s, with r, $s \geq 1$. A 3-dimensional generalization of this notion is now given by establishing the existence of an infinite family of vertex critical graphs each admitting three completely independent critical cliques. This confirms the existence of $\mathcal{K}$ for $|\mathcal{K}| = 3$.

4.1 The main construction

To begin the construction of such a family, select an arbitrary Type I matrix graph $G_1(A)$, where $A \in \mathcal{M}_n^3$. In what follows, the graph $G_1(A)$ will be referred to as the planet. Adjoin to the planet $G_1(A)$ three complete graphs of order $n-1$, say

$$K_{n-1}^{j}(s_j) = \left\{ x_{\widehat{\sigma}_1(s_j)}^{j,1}, x_{\widehat{\sigma}_2(s_j)}^{j,2}, \ldots, x_{\widehat{\sigma}_{n-1}(s_j)}^{j,n-1} \right\},$$

for $j = 1, 2, 3$. Here, an arbitrary 3-submultiset (or 3-combination) of I_n has been chosen, $\{s_1, s_2, s_3\}$, as well as the fixed permutation $\sigma = 1\,2\cdots n$ of I_n. These three complete graphs will be referred to as the satellites. The reason why such a peculiar notation for these satellites is needed will become clear below. As above, the family of satellites will be written as $\mathcal{K} = \left\{ K_{n-1}^{j}(s_j) : j = 1, 2, 3 \right\}$. Construct a graph G by defining $V(G)$ to be the set

$$V(G) = [V(G_1(A))] \cup [\cup\mathcal{K}]$$

and by defining $E(G)$ according to the following two prescriptions indicated below.

$$\begin{array}{ll} \text{(I)} & \text{For all } a_\alpha, a_\beta \in V(G_1(A)), \\ & a_\alpha a_\beta \in E(G) \text{ if and only if } a_\beta \notin \cup\mathcal{F}_{a_\alpha}. \\ & \\ \text{(II)} & \text{For all } x_k^{j,t} \in \cup\mathcal{K} \text{ and } a_\alpha \in V(G_1(A)), \\ & x_k^{j,t} a_\alpha \in E(G) \text{ if and only if } \alpha_p \neq k \text{ when } p \neq j. \end{array} \tag{6}$$

Example 3 *As an example to illustrate this labelling scheme, suppose that* $\{1,3,4\}$ *is chosen as the* 3*-submultiset of* I_4. *Also, let* $\sigma = 1\,2\,3\,4$ *so that for instance* $\widehat{\sigma}(2) = 1\,3\,4$. *Further, we have* $\widehat{\sigma}_1(2) = 1$, $\widehat{\sigma}_2(2) = 3$, *and* $\widehat{\sigma}_3(2) = 4$. *In this case, the vertices of the three satellites would be labelled as follows:*

$$\begin{aligned} K_3^1(1) &= \left\{ x_2^{1,1}, x_3^{1,2}, x_4^{1,3} \right\}, \\ K_3^2(3) &= \left\{ x_1^{2,1}, x_2^{2,2}, x_4^{2,3} \right\}, \end{aligned}$$

and

$$K_3^3(4) = \left\{ x_1^{3,1}, x_2^{3,2}, x_3^{3,3} \right\}.$$

Remark 3 *In (II) of* (6), *it is somewhat cumbersome to visualize how vertices in the planet are adjacent to vertices in the satellites. An equivalent formulation of the second prescription is the following:*

$$x_k^{j,t} a_\alpha \in E(G) \text{ if and only if } a_\alpha \in A^* \backslash \left(\bigcup_{i \neq j} F_{ik}(A) \right).$$

In other words, connect the vertex $x_k^{j,t}$ *from the satellite* $K_{n-1}^{j}(s_j)$ *to all vertices that remain in the planet* $G_1(A)$ *after removing the* k*th face of* A *in the* i*th direction for all directions except the* j*th direction. Figures 11-13 should make this more clear.*

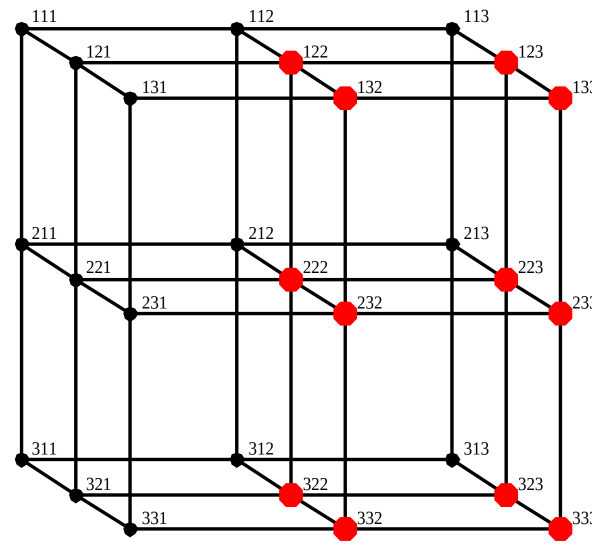

Figure 11: Vertices adjacent to $x_1^{1,1}$.

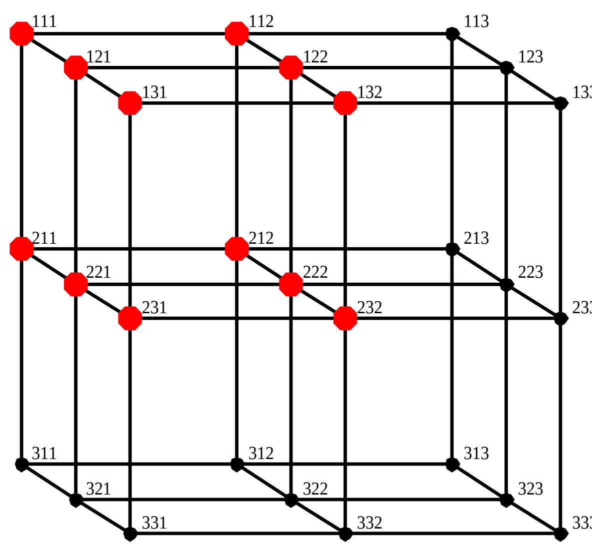

Figure 12: Vertices adjacent to $x_3^{2,1}$.

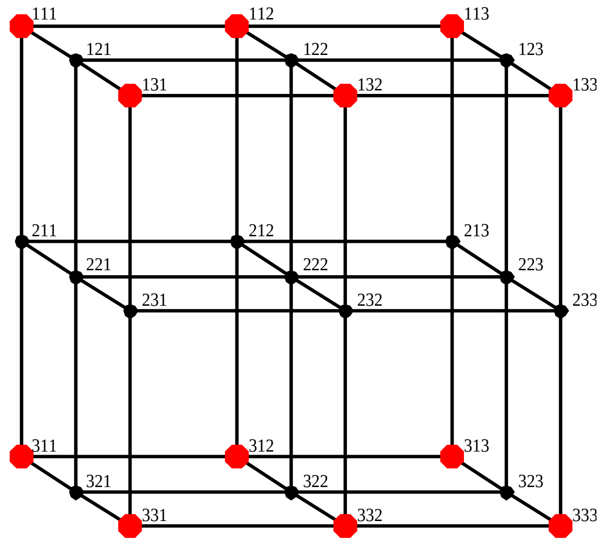

Figure 13: Vertices adjacent to $x_2^{3,1}$.

In order to underscore the dependence of G on the dimensions associated with the underlying k-dimensional n-square matrix graph and the set of satellites, we shall write $G = G_1^{k,\,n}(A, \mathcal{K})$. Here, A denotes the associated k-dimensional n-square matrix and $\mathcal{K}$ represents the set of satellites, each member of which is a complete graph of order $n-1$, adjoined to the planet $G_1(A)$. Also, observe now that the choice for the notation in the labelling of the vertices of each satellite is necessary as it provides a convenient method for describing exactly how the satellites are attached to the planet. The first objective is to determine $\chi\left(G_1^{3,\,n}(A, \mathcal{K})\right)$.

Theorem 3 *The graph $G_1^{3,\,n}(A, \mathcal{K})$ is $(2n-1)$-chromatic for every $n \geq 3$.*

Proof. From Theorem 2 above, it follows that $\chi(G_1(A)) = n$. Clearly, $\chi\left(K_{n-1}^{j}(s_j)\right) = n-1$. When the satellites are adjoined to form the graph $G_1^{3,\,n}(A, \mathcal{K})$, we claim that $\chi\left(G_1^{3,\,n}(A, \mathcal{K})\right) = n+(n-1) = 2n-1$.

The proof proceeds by induction on n. For the base case of $n=3$, consider the graph $G_1^{3,\,3}(A, \mathcal{K})$. It must be shown that $\chi\left(G_1^{3,\,3}(A, \mathcal{K})\right) = 5$. By the definition of $G_1^{3,\,3}(A, \mathcal{K})$, the planet $G_1(A)$ is a Type I matrix graph with $A \in \mathcal{M}_3^3$ and so $\chi(G_1(A)) = 3$. Because each satellite is a 2-clique and there are no edges among distinct satellites by the definition of the edge set $E\left(G_1^{3,\,3}(A, \mathcal{K})\right)$ in (6), it is clear that the chromatic number of each satellite is 2 and consequently, $\chi\left(G_1^{3,\,3}(A, \mathcal{K})\right) \leq 5$. It remains to show that $\chi\left(G_1^{3,\,3}(A, \mathcal{K})\right) \geq 5$. To this end, consider an arbitrary partition $\mathcal{P}$ of $V\left(G_1^{3,\,3}(A, \mathcal{K})\right)$ into a minimal number of independent subsets. Every $X \in \mathcal{P}$ has the form $X = X_S \cup X_P$, where X_S is an independent subset of $\cup\mathcal{K}$, the set of all satellite vertices; and, X_P is an independent subset of A^*, the set of all planet vertices. Up to automorphism, there are exactly three distinct patterns as to how the vertices in the satellites can be adjacent to vertices in the planet. They are distinguished by the repetition numbers of the associated submultiset of I_3. The submultiset that determines the pattern is given in parentheses.

Table 1: Pattern I ({3, 3, 3})

	X_S	Y_S	Z_S
$K_2^1\,(3)$:	$x_1^{1,\,1}$	$x_2^{1,\,2}$	
$K_2^2\,(3)$:	$x_1^{2,\,1}$	$x_2^{2,\,2}$	
$K_2^3\,(3)$:	$x_1^{3,\,1}$	$x_2^{3,\,2}$	

Table 2: Pattern II ({3, 3, 2})

	X_S	Y_S	Z_S
$K_2^1\,(3)$:	$x_1^{1,\,1}$	$x_2^{1,\,2}$	
$K_2^2\,(3)$:	$x_1^{2,\,1}$	$x_2^{2,\,2}$	
$K_2^3\,(2)$:	$x_1^{3,\,1}$		$x_3^{3,\,2}$

Table 3: Pattern III ({3, 2, 1})

	X_S	Y_S	Z_S
$K_2^1\,(3)$:	$x_1^{1,\,1}$	$x_2^{1,\,2}$	
$K_2^2\,(2)$:	$x_1^{2,\,1}$		$x_3^{2,\,2}$
$K_2^3\,(1)$:		$x_2^{3,\,1}$	$x_3^{3,\,2}$

In each pattern, the vertices of the satellites are represented in the rows of the table and the vertices in the columns will be grouped in various ways to represent how satellite vertices are distributed among the independent subsets contained in $\mathcal{P}$.

Now for each of these three patterns, there are two ways in which vertices in the columns can be colored; i.e., distributed among independent subsets of $\mathcal{P}$. These ways are indicated in (A) and (B) below.

(A) For each column, all vertices in the column are in a single color class.
(B) There exists a column for which not all of the vertices in the column are contained in a single color class.

For instance, in Pattern II, if the first type of coloration as in (A) is illustrated, then we would have

$$\begin{aligned} X &= X_S \cup X_P = \left\{x_1^{1,1},\, x_1^{2,1},\, x_1^{3,1}\right\} \cup X_P \\ Y &= Y_S \cup Y_P = \left\{x_2^{1,2},\, x_2^{2,2}\right\} \cup Y_P \\ Z &= Z_S \cup Z_P = \left\{x_3^{3,2}\right\} \cup Z_P \end{aligned}$$

as three representative elements from $\mathcal{P}$. Also, in Pattern I, if the second type of coloration as in (B) is illustrated, then we would have

$$\begin{aligned} X &= X_S \cup X_P = \left\{x_1^{1,1}\right\} \cup X_P \\ X' &= X'_S \cup X'_P = \left\{x_1^{2,1},\, x_1^{3,1}\right\} \cup X'_P \\ Y &= Y_S \cup Y_P = \left\{x_2^{1,2},\, x_2^{2,2},\, x_2^{3,2}\right\} \cup Y_P \end{aligned}$$

as three representative elements from $\mathcal{P}$. Observe that all vertices in the first column of Pattern I are shared (distributed) among two distinct color classes in $\mathcal{P}$.

Consider the first type of coloration given by (A) for each pattern separately. For Pattern I, the arbitrary partition $\mathcal{P}$ contains elements X and Y where

$$X = \left\{x_1^{1,1},\, x_1^{2,1},\, x_1^{3,1}\right\} \cup X_P$$

and

$$Y = \left\{x_2^{1,2},\, x_2^{2,2},\, x_2^{3,2}\right\} \cup Y_P.$$

Now consider the graph $G_1^{3,3}(A, \mathcal{K}) - X - Y$. To prove there are at least three elements of $\mathcal{P} \setminus \{X, Y\}$, it suffices to prove that there exists a subgraph of $G_1^{3,3}(A, \mathcal{K}) - X - Y$ that is isomorphic to K_3. To see that this is indeed the case, consider the set $W = \{a_{123}, a_{231}, a_{312}\}$. By Proposition 9, it follows that $G_1^{3,3}(A, \mathcal{K})[W] \cong K_3$. Moreover, $W \subseteq V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$ since by the definition of the edge set of $G_1^{3,n}(A, \mathcal{K})$, it must be that

$$x_1^{1,1}a_{123},\, x_1^{3,1}a_{231},\, x_1^{2,1}a_{312} \in E\left(G_1^{3,n}(A, \mathcal{K})\right)$$

and

$$x_2^{2,2}a_{123},\, x_2^{1,2}a_{231},\, x_2^{3,2}a_{312} \in E\left(G_1^{3,n}(A, \mathcal{K})\right).$$

Therefore, there must be at least three elements of the partition $\mathcal{P}$ that remain upon the removal of X and Y from $\mathcal{P}$.

In the next case, consider Pattern II. Here, the arbitrary partition $\mathcal{P}$ contains elements X, Y, and Z, where

$$X = \left\{x_1^{1,1},\, x_1^{2,1}x_1^{3,1}\right\} \cup X_P,$$

$$Y = \left\{x_2^{1,2},\, x_2^{2,2}\right\} \cup Y_P,$$

and

$$Z = \left\{x_3^{3,2}\right\} \cup Z_P.$$

As above, consider the graph $G_1^{3,3}(A, \mathcal{K}) - X - Y$. Clearly, either $a_{132} \in V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$ or $a_{221} \in V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$. This is because

$$x_1^{1,1}a_{132},\, x_1^{3,1}a_{221},\, a_{132}a_{221} \in E\left(G_1^{3,n}(A, \mathcal{K})\right).$$

In the event that $a_{132} \in V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$, the set $W_1 = \{a_{132}, a_{213}, a_{321}\}$ satisfies

$$W_1 \subseteq V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$$

and

$$G_1^{3,3}(A, \mathcal{K})[W_1] \cong K_3.$$

And in the event that $a_{221} \in V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$, set $W_2 = \left\{a_{221}, x_3^{3,2}, a_{123}, a_{231}, a_{313}\right\}$ satisfies

$$W_2 \subseteq V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y\right)$$

and

$$G_1^{3,3}(A, \mathcal{K})[W_2] \cong C_5,$$

the cycle on five vertices. In either case, there must exist at least three elements of $\mathcal{P}$ that remain upon the removal of X and Y from $\mathcal{P}$.

In the last case, consider Pattern III. The arbitrary partition $\mathcal{P}$ contains elements X, Y, and Z, where

$$X = \left\{x_1^{1,1}, x_1^{2,1}\right\} \cup X_P,$$

$$Y = \left\{x_2^{1,2}, x_2^{3,1}\right\} \cup Y_P,$$

and

$$Z = \left\{x_2^{2,2}, x_3^{3,2}\right\} \cup Z_P.$$

Now, if these three elements of $\mathcal{P}$ are removed, then the set $W_3 = \{a_{132}, a_{213}\}$ satisfies

$$W_3 \subseteq V\left(G_1^{3,3}(A, \mathcal{K}) - X - Y - Z\right)$$

and

$$G_1^{3,3}(A, \mathcal{K})[W_3] \cong K_2,$$

implying that there are at least five members of $\mathcal{P}$. Therefore, if the type of coloration in (A) is considered, it follows that $\chi\left(G_1^{3,3}(A, \mathcal{K})\right) \geq 5$. Consequently, in this case, $\chi\left(G_1^{3,3}(A, \mathcal{K})\right) = 5$.

Suppose now that the latter type of coloration in (B) is implemented. Then it is no longer necessary to consider separately the three patterns of adjacencies described above. Assume there exists a column in which there are two vertices in distinct satellites that are in distinct color classes. Call these vertices $x_k^{j_1,t_1}$ and $x_k^{j_2,t_2}$ and suppose they belong to the color classes X and Y, respectively. Upon the removal of the color classes X and Y from the partition $\mathcal{P}$, there is a subgraph of $G_1^{3,3}(A, \mathcal{K}) - X - Y$, call it H, such that

H is isomorphic to a graph of the form $G_1^{3,2}(B, \mathcal{K}')$, where $B \in \mathcal{M}_2^3$ and $\mathcal{K}'$ is family of 1-cliques. Note that in general, the eight elements of B will be the vertices in the set

$$Z = \left[A^* \setminus \left(\bigcup_{i \neq j_1} F_{ik}(A)\right)\right] \cap \left[A^* \setminus \left(\bigcup_{i \neq j_2} F_{ik}(A)\right)\right].$$

The six edges between the satellites and planet of $G_1^{3,2}(B, \mathcal{K}')$ can be selected from the up to eight remaining edges between the vertices that remain in the satellites and the planet of $G_1^{3,3}(A, \mathcal{K})$ when $x_k^{j_1, t_1}$, $x_k^{j_2, t_2}$, and $x_k^{j_3, t_0}$ are removed from the set of satellite vertices of $\cup\mathcal{K}$, where $j_3 \in I_3 \setminus \{j_1, j_2\}$ and for some $t_0 \in \{1, 2\}$. However, the edges that remain between the satellites and the planets do not play a role in this portion of the proof. The only thing that matters is how the eight vertices in Z are distributed among the color classes contained in $\mathcal{P}$. Furthermore, although the graph $G_1^{3,2}(B, \mathcal{K}')$ will not in general be part of the family of interest since $2 < 3$, it is constructed in the same fashion as when $n \geq k$. This is the only time that a graph of the form $G_1^{3,2}(B, \mathcal{K}')$ will be considered in this paper.

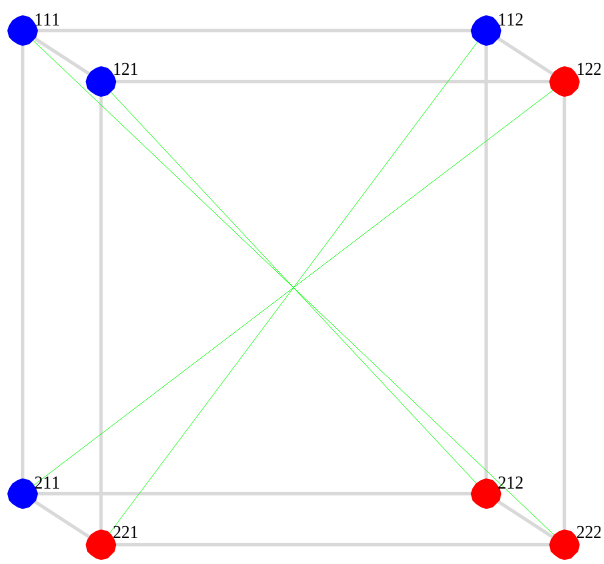

Figure 14: The planet $G_1(B)$ of H.

There are exactly three colorations, up to automorphism, of the planet $G_1(B)$ shown in Figure 14 above. The light gray lines are for visual purposes only. However, it suffices to consider only two types of colorations. These ways are indicated in (C) and (D) as follows:

(C) Three colors are used to color $G_1(B)$
(D) Two colors are used to color $G_1(B)$.

In the event that three colors have been used to color $G_1(B)$ as in (C), it is then clear that at least five colors have been used in the coloration of $G_1^{3,3}(A, \mathcal{K})$. On the other hand, in the event that only two colors have been used to color the graph $G_1(B)$ as in (D), we assert that five colors must be used to color just the planet $G_1(A)$ itself. To prove this assertion, it suffices to exhibit three vertices of $G_1(A)$ such that the subgraph of $G_1(A)$ induced by this set of three vertices is isomorphic to K_3; and, that each of these three vertices is adjacent to at least one vertex from each color class in the two coloring of $G_1(B)$ being considered in (D). In the Figure 15 below, we have embedded an isomorphic copy of $G_1(B)$ in the graph $G_1(A)$. The light gray lines are for visual purposes only and the gray lines demonstrate a subgraph isomorphic to K_3 induced by the set of vertices that colored green, yellow, and coral.

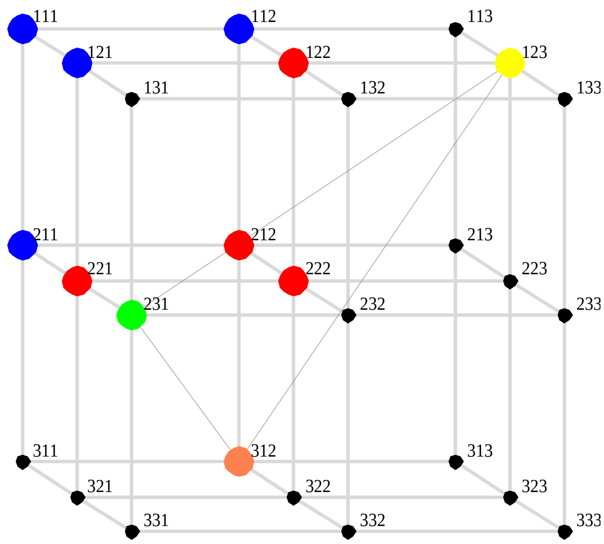

Figure 15: Vertices of an induced K_3.
$\{a_{123},\, a_{231},\, a_{312}\}$

The three vertices are selected as follows. Choose a representative 3-matching normal basis such that each normal contains two differently colored vertices from the embedded 3-dimensional 2-square matrix graph. There will be one vertex remaining in each normal. These three vertices are the ones sought. (What we mean is this: Consider the light gray lines in the embedded planet $G_1(B)$. Choose a set of three normals in $G_1(B)$ such that the two vertices in each normal are colored differently and that the three normals form a 3-matching in the planet $G_1(B)$. Now consider the extension of these three normals to three normals in the planet $G_1(A)$. Select one vertex from each of these three normals in $G_1(A)$ that was not a vertex in the corresponding normal in $G_1(B)$.) By Proposition 9, these three vertices determine an induced subgraph isomorphic to K_3. This proves that $\chi\left(G_1^{3,\,3}(A,\,\mathcal{K})\right) = 5$ and establishes the base case.

Inductively assume that $\chi\left(G_1^{3,\,r}(A,\,\mathcal{K})\right) = 2r-1$ and consider the graph $G_1^{3,\,r+1}(A,\,\mathcal{K})$. Let $\mathcal{P}$ be an arbitrary partition of $G_1^{3,\,r+1}(A,\,\mathcal{K})$ into a minimal number of independent subsets. As in the base case, there are exactly three ways, up to automorphism, that vertices in the satellites can be adjacent to vertices in the planet. These patterns are indicated below.

Table 4: Pattern I ({3, 3, 3})

	$X_{S,1}$	$X_{S,2}$	$X_{S,3}$	$X_{S,4}$	$\cdots$	$X_{S,r}$	$X_{S,r+1}$
$K_2^1\,(3)$:	$x_1^{1,1}$	$x_2^{1,2}$		$x_4^{1,3}$	$\cdots$	$x_r^{1,r-1}$	$x_{r+1}^{1,r}$
$K_2^2\,(3)$:	$x_1^{2,1}$	$x_2^{2,2}$		$x_4^{2,3}$	$\cdots$	$x_r^{2,r-1}$	$x_{r+1}^{2,r}$
$K_2^3\,(3)$:	$x_1^{3,1}$	$x_2^{3,2}$		$x_4^{3,3}$	$\cdots$	$x_r^{3,r-1}$	$x_{r+1}^{3,r}$

Table 5: Pattern II ({3, 3, 2})

	$X_{S,1}$	$X_{S,2}$	$X_{S,3}$	$X_{S,4}$	$\cdots$	$X_{S,r}$	$X_{S,r+1}$
$K_2^1\,(3)$	$x_1^{1,1}$	$x_2^{1,2}$		$x_4^{1,3}$	$\cdots$	$x_r^{1,r-1}$	$x_{r+1}^{1,r}$
$K_2^2\,(3)$	$x_1^{2,1}$	$x_2^{2,2}$		$x_4^{2,3}$	$\cdots$	$x_r^{2,r-1}$	$x_{r+1}^{2,r}$
$K_2^3\,(2)$	$x_1^{3,1}$		$x_3^{3,2}$	$x_4^{3,3}$	$\cdots$	$x_r^{3,r-1}$	$x_{r+1}^{3,r}$

Table 6: Pattern III ({3, 2, 1})

	$X_{S,1}$	$X_{S,2}$	$X_{S,3}$	$X_{S,4}$	$\cdots$	$X_{S,r}$	$X_{S,r+1}$
$K_2^1\,(3)$:	$x_1^{1,1}$	$x_2^{1,2}$		$x_4^{1,3}$	$\cdots$	$x_r^{1,r-1}$	$x_{r+1}^{1,r}$
$K_2^2\,(2)$:	$x_1^{2,1}$		$x_3^{2,2}$	$x_4^{2,4}$	$\cdots$	$x_r^{2,r}$	$x_{r+1}^{2,r}$
$K_2^3\,(1)$:		$x_2^{3,1}$	$x_3^{3,2}$	$x_4^{3,3}$	$\cdots$	$x_r^{3,r-1}$	$x_{r+1}^{3,r}$

Now for each of these three patterns, there are two ways in which vertices in the columns can be colored; i.e., distributed among independent subsets of $V\left(G_1^{3,3}(A,\mathcal{K})\right)$. These ways are exactly as they were in the base case above. The are restated for convenience.

A. For each column, all vertices in the column are in a single color class.
B. There exists a column for which not all of the vertices in the column are contained in a single color class.

Let us assume the former coloration in (A) is implemented. For Pattern I, after r elements of the partition $\mathcal{P}$ are removed, the vertices in the set

$$X = \left\{a_{(1,2,3)},\, a_{(2,3,4)},\, \ldots,\, a_{(r,r+1,1)},\, a_{(r+1,1,2)}\right\}$$

determine, by Proposition 9, an induced subgraph $G[X]$ satisfying the isomorphism $G[X] \cong K_{r+1}$. The fact that these vertices do in fact remain, is confirmed by the chart of adjacencies below. In the chart, a planet vertex appearing in column j is adjacent to every satellite vertex appearing in column j.

Table 7: Adjacencies in $G_1(A)$ between the satellite vertices and planet vertices

	$a_{(1,2,3)}$	$a_{(2,3,4)}$	$\cdots$	$a_{(r,r+1,1)}$	$a_{(r+1,1,2)}$
$X_{S,1}$	$x_1^{1,1}$	$x_1^{2,1}$	$\cdots$	$x_1^{3,1}$	$x_1^{2,1}$
$X_{S,2}$	$x_2^{2,2}$	$x_2^{1,2}$	$\cdots$	$x_2^{1,2}$	$x_2^{3,2}$
$X_{S,3}$					
$X_{S,4}$	$x_4^{3,3}$	$x_4^{3,3}$	$\cdots$	$x_4^{2,3}$; if $r=3$ $x_4^{1,3}$; if $r=4$ any ; otherwise	$x_4^{1,3}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$X_{S,r}$	$x_r^{3,r-1}$	$x_r^{3,r-1}$	$\cdots$	$x_r^{1,r-1}$	$x_r^{2,r-1}$
$X_{S,r+1}$	$x_{r+1}^{3,r}$	$x_{r+1}^{3,r}$	$\cdots$	$x_{r+1}^{2,r}$	$x_{r+1}^{1,r}$

Therefore, we see that there must be at least $r+1$ elements of $\mathcal{P}$ that remain upon the removal of the color r classes $X_{S,1}, X_{S,2}, X_{S,4}, X_{S,5}, \ldots, X_{S,r}, X_{S,r+1}$.

For Pattern II, after removing up to $r+1$ elements from the partition $\mathcal{P}$, we have three subcases to consider.

Case 1 (Coloration A, Pattern II) The equation $r+1=4$ is true so that $r=3$.

In this case, the set

$$X = \{a_{124},\, a_{241},\, a_{413}\}$$

satisfies $G_1^{3,\,r+1}(A,\,\mathcal{K})[X] \cong K_3$ by Proposition 9 so that

$$\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) = 4+3 = 7 = 2(r+1)-1.$$

Case 2 (Coloration A, Pattern II) The equation $r+1=5$ is true so that $r=4$.

In this case, the set

$$X = \{a_{123},\, a_{245},\, a_{541},\, a_{514}\}$$

satisfies $G_1^{3,\,r+1}(A,\,\mathcal{K})[X] \cong K_4$ by Proposition 9 so that

$$\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) = 5+4 = 9 = 2(r+1)-1.$$

Case 3 (Coloration A, Pattern II) The inequality $r+1>5$ is true so that $r>4$.

In the last case, after removing the $r-2$ elements $X_4, X_5, \ldots, X_r, X_{r+1}$, the set

$$X = \left\{a_{(4,\,5,\,6)},\, a_{(5,\,6,\,7)},\, \ldots,\, a_{(r-1,\,r,\,r+1)},\, a_{(r,r+1,\,4)},\, a_{(r+1,\,4,\,5)}\right\}$$

satisfies $G_1^{3,\,r+1}(A,\,\mathcal{K})[X] \cong K_{r-2}$ by Proposition 9 so that, in addition to the subgraph H defined above which requires 5 colors, we have

$$\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) = (r-2)+(r-2)+5 = 2(r+1)-1.$$

For Pattern III, observe that if $r+1>5$, then this pattern would be the same as Pattern II. Hence, we may assume that $r+1 \le 5$. There are two cases to consider after removing all $r+1$ partition elements.

Case 1 (Coloration A, Pattern III) The equation $r+1=4$ is true so that $r=3$.

In this case, the set

$$X = \{a_{134},\, a_{243},\, a_{412}\}$$

satisfies $G_1^{3,\,r+1}(A,\,\mathcal{K})[X] \cong K_3$ by Proposition 9 so that

$$\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) = 4+3 = 7 = 2(r+1)-1.$$

Case 2 (Coloration A, Pattern III) The equation $r+1=5$ is true so that $r=4$.

In this case, the set

$$X = \{a_{134},\, a_{245},\, a_{452},\, a_{513}\}$$

satisfies $G_1^{3,\,r+1}(A,\,\mathcal{K})[X] \cong K_4$ by Proposition 9 so that

$$\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) = 5+4 = 9 = 2(r+1)-1.$$

Suppose now that the latter type of coloration is implemented. In this event, it no longer becomes necessary to consider separately the three patterns of adjacencies described above. Assume that there exists a column in which there are two vertices in distinct satellites that are in distinct color classes. Call these vertices $x_k^{j_1,\,t_1}$ and $x_k^{j_2,\,t_2}$. Moreover, suppose these two vertices belong to the color classes X and Y, respectively. Upon the removal of the color classes X and Y from the partition $\mathcal{P}$, we assert that there exists a subgraph H of $G_1^{3,\,r+1}(A,\,\mathcal{K})$, in the remaining subgraph, that is isomorphic to $G_1^{3,\,r}(A,\,\mathcal{K})$. To see this, we consider the following two cases.

Case 1 (Coloration B) When the color classes X and Y are removed from $\mathcal{P}$, at most one vertex from each satellite is removed. In this case, $r-1$ vertices remain in each clique when X and Y are removed. Therefore the is a subgraph H of $G_1^{3,\,r+1}(A,\,\mathcal{K})$ that is isomorphic to $G_1^{3,\,r}(A,\,\mathcal{K})$.

Case 2 (Coloration B) When the color classes X and Y are removed from $\mathcal{P}$, there exists a satellite for which two vertices have been removed. Note that there cannot be more than two vertices removed from any satellite when X and Y are removed from $\mathcal{P}$. For each such satellite that have two vertices removed, there is a vertex that can be taken from the planet and moved into orbit to serve as a new satellite vertex. If this can be established for each satellite that had two vertices removed, then the previous case applies, in which each satellite would have $r-1$ vertices. Suppose that $K_r^j(s_j)$ is a satellite that had two vertices removed. Recall that

$$K_r^j(s_j) = \left\{x^{j,\,1}_{\widehat{\sigma}_1(s_j)},\, x^{j,\,2}_{\widehat{\sigma}_2(s_j)},\, \ldots,\, x^{j,\,r}_{\widehat{\sigma}_r(s_j)}\right\}.$$

Further assume that vertices

$$x^{j,\,n_1}_{\widehat{\sigma}_{n_1}(s_j)}$$

and

$$x^{j,\,n_2}_{\widehat{\sigma}_{n_2}(s_j)}$$

have been removed upon the removal of X and Y. Define $\beta(j) \in I_n^k$ by the rule

$$(\beta(j))_i = \begin{cases} j & \text{if } i = j \\ s_j & \text{if } i \neq j \end{cases}.$$

By the definitions of the satellite $K_r^j(s_j)$ and the edge set $G_1^{3,\,n}(A,\,\mathcal{K})$, the vertex $a_{\beta(j)}$ is adjacent to every vertex in $K_r^j(s_j)$. Therefore, $a_{\beta(j)}$ remains upon the removal of X and Y. Thus, the new satellite

$$\left(\left\{a_{\beta(j)}\right\} \cup K_r^j(s_j)\right) \setminus (X \cup Y)$$

has order $r-1$. We remark that it is not required that $a_{\beta(j)}$ be independent from the other satellites. This is because we are only interested in finding a subgraph H that is isomorphic to $G_1^{3,\,r}(A,\,\mathcal{K})$.

Now, in the case of Coloration B, we have shown that in the subgraph $G_1^{3,\,r+1}(A,\,\mathcal{K}) - X - Y$, there exists a subgraph H of satisfying $H \cong G_1^{3,\,r}(A,\,\mathcal{K})$. Therefore, by the inductive hypothesis,

$$\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K}) - (X \cup Y)\right) \geq \chi\left(G_1^{3,\,r}(A,\,\mathcal{K})\right) = 2r-1.$$

Consequently, $\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) \geq (2r-1)+2 = 2r+1$ and since one can easily exhibit a $(2r+1)$-coloring of $G_1^{3,\,r+1}(A,\,\mathcal{K})$ by coloring the planet with $r+1$ colors and coloring the satellites with an additional r colors, it follows that $\chi\left(G_1^{3,\,r+1}(A,\,\mathcal{K})\right) = 2r+1$. By the principle of mathematical induction, we conclude that $\chi\left(G_1^{3,\,n}(A,\,\mathcal{K})\right) = 2n-1$ for all $n \geq 3$. ■

Theorem 4 *The set* $\mathcal{G} = \left\{G_1^{3,\,n}(A,\,\mathcal{K}) : n \geq 3\right\}$ *is a family of graphs satisfying the following conditions.*

1. $\chi\left(G_1^{3,\,n}(A,\,\mathcal{K})\right) = 2n-1.$
2. *The induced subgraphs* $G_1^{3,\,n}(A,\,\mathcal{K})\left[K_{n-1}^{j}(s_j)\right]$, *for* $j = 1, 2, 3$, *are completely independent critical* $(n-1)$*-cliques.*

Proof. *By Theorem 3 , the graph* $G_1^{3,\,n}(A,\,\mathcal{K})$ *is* $(2n-1)$*-chromatic. For any direction* i, *where* $1 \leq i \leq 3$, *color the* j*th face of* A *in the* i*th direction with color* c_j *for* $j \in I_n$. *Necessarily, the satellite* $K_{n-1}^{i}(s_i)$ *would require* $n-1$ *additional colors, say* $c_{n+1},\, c_{n+2},\, \ldots,\, c_{2n-1}$, *while the remaining satellites can be colored from among the colors* $c_1,\, c_2,\, \ldots,\, c_n$. *This coloration shows that the satellite* $G_1^{3,\,n}(A,\,\mathcal{K})\left[K_{n-1}^{i}(s_i)\right]$ *is a critical* $(n-1)$*-clique. By the definition of* $E\left(G_1^{3,\,n}(A,\,\mathcal{K})\right)$, *these cliques are also completely independent.* ■

Corollary 5 *For every* $p,\, q,\, r \in \mathbb{N}$, *where* $p,\, q,\, r \geq 2$, *there exists a vertex critical graph which admits critical cliques having orders* p, q, *and* r *that are completely independent.*

5 Concluding Remarks

It seems as though graphs admitting completely independent critical cliques are rare. It would be interesting to determine whether or not there are other families of graphs admitting completely independent critical cliques; or to be able to classify graphs which have this property. Also, what condition can be imposed on a graph to guarantee that a certain number of edges between critical cliques must exist? The answer to this question might lead to answering double-critical conjecture of Lovász in the affirmative. The generalization from $G_1^{3,\,n}(A,\,\mathcal{K})$ to $G_{k,\,n}(A,\,\mathcal{K})$ at present is incomplete but will be investigated further. We conclude by with the following conjecture.

Conjecture 1 *For every pair* $m,\, n \in \mathbb{N}$, *there exists a vertex critical graph admitting* m *completely independent critical cliques of order* n.

Thus far, this conjecture holds for all n and $m = 1$, 2, and 3.

References

[1] J. Balogh, A.V. Kostochka, N. Prince, and M. Stiebitz, The Erdős-Lovász Tihany conjecture for quasi-line graphs, *Discrete Math.*, **309** (12) (2009), pp. 3985-3991. doi: 10.1016/j.disc.2008.11.016

[2] R.A. Brualdi, Introductory Combinatorics, 5th ed, Pearson, Upper Saddle River (2010)

[3] G. Chartrand, L. Lesniak, and P. Zhang, Graphs and Digraphs, 5th ed, CRC Press, Boca Raton (2010)

[4] G.A. Dirac, A theorem of R.L. Brooks and a conjecture of H. Hadwiger, *Proc. London Math. Soc.* (3) **7** (1957), pp. 161-195.

[5] G.A. Dirac, The number of edges in critical graphs, *J. Reine Angew. Math.* **268/269** (1974), pp. 150-164.

[6] P. Erdős, *Problems*, *Theory of Graphs*, *Proc. Colloq.* Tihany, Academic Press, New York (1968), pp. 361-362.

[7] T.R. Jensen, Dense critical and vertex-critical graphs, *Discrete Math.*, **258** (2002), pp. 63-84.

[8] T.R. Jensen and B. Toft, Graph Coloring Problems, Wiley-Interscience, New York (1995).

[9] K-i. Kawarabayashi, A. Sune Pedersen, B. Toft, Double-critical graphs and complete minors, Retrieved from http://adsabs.harvard.edu/abs/2008arXiv0810.3133K

[10] A.V. Kostochka and M. Stiebitz, Colour-critical graphs with few edges, *Discrete Math.*, **191** (1998), pp.125-137.

[11] A.V. Kostochka and M. Stiebitz, On the number of edges in colour-critical graphs and hypergraphs, *Combinatorica* **20** (2000), pp. 521-530.

[12] J.J. Lattanzio, Completely independent critical cliques, *J. of Comb. Math. and Comb. Comp.* **62** (2007), pp. 165-170.

[13] J.J. Lattanzio, Edge Double-Critical Graphs, J. Math. & Stat., **6** (3) (2010), pp. 357-358. Retrieved from http://www.scipub.org/fulltext/jms2/jms263357-358.pdf

[14] M. Stiebitz, K_5 is the only double-critical 5-chromatic graph, *Discrete Math.* **64** (1987), pp. 91-93.

[15] B. Toft, On the maximal number of edges of critical k-chromatic graphs, *Studia Sci. Math. Hungar.* **5** (1970), pp. 461-470.