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SOME FACINATING THEOREMS OF THE MULATU NUMBERS

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Some Fascinating Theorems of the Mulatu Numbers

Scholarship of Teaching and Learning paper

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Abstract: The Mulatu numbers were introduced in [1] . The numbers are sequences of numbers of the form: **4,1, 5,6,11,17,28,45...** The numbers have wonderful and amazing properties and patterns.

In mathematical terms, the sequence of the Mulatu numbers is defined by the following recurrence relation:

$$M_n := \begin{cases} 4 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ M_{n-1} + M_{n-2} & \text{if } n > 1. \end{cases}$$

The double Angel Formulas for Fibonacci and Lucas numbers are given by the following formulas respectively**.**

(1) $F_{2n} = F_n L_n$ **and** (2) $L_{2n} = \dfrac{5F^2{}_n + L^2{}_n}{2}$

Since both the Fibonacci and Lucas numbers have double angle Formulas, It is natural to ask if such formula exist for Mulatu Numbers. The answer is affirmative and produces the following paper.

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Key Words: Mulatu numbers, Mulatu sequences, Fibonacci numbers, Lucas numbers, Fibonacci sequences, and Lucas sequences.

1. **<u>Introduction and Background</u>**. As given in [1], the Mulatu numbers are a sequence of numbers recently introduced by Mulatu Lemma, Professor of

Mathematics at Savannah State University, Savannah, Georgia, USA. The Mulatu sequence has wealthy mathematical properties and patterns like the two celebrity sequences of Fibonacci and Lucas.

In this paper, more interesting relationships of the Mulatu numbers to the Fibonacci and Lucas numbers will be presented.

Here are the First 21 Mulatu, Fibonacci, and Lucas numbers for quick reference.

Mulatu(M_n), **Fibonacci**(F_n) and **Lucas**(L_n) **Numbers**
(Tables 1 & 2)

Table 1

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
M_n:	4	1	5	6	11	17	28	45	73	118	191	309
F_n:	0	1	1	2	3	5	8	13	21	34	55	89
L_n:	2	1	3	4	7	11	18	29	47	76	123	199

Table 2

n:	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
M_n	500	809	1309	2118	3427	5545	8972	14517	23489
F_n:	144	233	377	610	987	1597	2584	4181	6765
L_n:	322	521	843	1364	2207	3571	5778	9349	15127

Remark 1 : Throughout this paper M, F, and L stand for Mulatu numbers, Fibonacci numbers, and Lucas number respectively.

The following well-known identities of Mulatu numbers, Fibonacci numbers, and Lucas numbers are required in this paper and hereby listed for quick reference.

(1) $L_n = F_{n-1} + F_{n+1}$

(2) $F_{n+1} = F_n + F_{n-1}$

(3) $F_{2n} = F_n L_n$

(4) $L_{2n} = F_n + 2F_{n-1}$

(5) $F_n = \dfrac{L_{n+1} + L_{n-1}}{5}$

(6) $L_{n+1} = L_n + L_{n-1}$

(7) $F_{n+k} = F_{n-1}F_k + F_nF_{k+1}$

(8) $5F^2{}_n - L^2{}_n = 4(-1)^{n+1}$

(9) $L_{n+m} = \dfrac{5F_nF_m + L_nL_m}{2}$

(10) $M_{n+k} = F_{n-1}M_k + M_{k+1}F_n$

The Main Results.

We will state the following theorem proved in [1] as proposition 1 and use it.

Proposition 1. $M_n = F_{n-3} + F_{n-1} + F_{n+2}$

Theorem 1: The following are equivalent.

(1) M_n

(2) $F_{n-3} + F_{n-1} + F_{n+2}$

(3) $L_n + 2F_{n-1}$

(4) $F_n + 4F_{n-1}$

(5) $4F_{n+1} - 3F_n$

Proof: We will show that (1)$\Rightarrow$ (2) $\Rightarrow$(3) $\Rightarrow$(4) $\Rightarrow$(5) $\Rightarrow$(1)

(i) (1)$\Rightarrow$ (2**)** follow**s** by Proposition 1.

(ii) (2))$\Rightarrow$(3) follows as shown:

$$F_{n-3} + F_{n-1} + F_{n+2} = F_{n-3} + F_{n-1} + F_{n+1} + F_n$$
$$= F_{n-3} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2}$$
$$= F_{n-1} - F_{n-2} + F_{n-1} + F_{n+1} + F_{n-1} + F_{n-2}$$
$$= 2F_{n-1} + L_n$$

(iii)(3) $\Rightarrow$(4) follows as shown:

$$L_n + 2F_{n-1} = F_{n+1} + F_{n-1} + 2F_{n-1}$$
$$= F_n + F_{n-1} + F_{n-1} + 2F_{n-1}$$
$$= F_n + 4F_{n-1}$$

(iv) (4) $\Rightarrow$(5) follows as shown:

$$F_n + 4F_{n-1} = F_n + 4(F_{n+1} - F_n)$$
$$= 4F_{n+1} - 3F_n$$

(v) (5) $\Rightarrow$(1) follows as shown:

$$4F_{n+1} - 3F_n = 4F_{n+1} - 3(F_{n+1} - F_{n-1}) = F_{n+1} + 3F_{n-1} = F_{n+1} + F_{n-1} + F_{n-1} + F_{n-1}$$
$$= F_{n+1} + (F_n - F_{n-2}) + F_{n-1} + F_{n-3} + F_{n-2} = F_{n+1} + F_n - F_{n-2} + F_{n-1} + F_{n-3} + F_{n-2}$$

$= F_{n+2} + F_{n-1} + F_{n-3} = M_n$ by Proposition 1 and hence) (5) $\Rightarrow$(1). Thus the theorem is proved.

Theorem 2.

$$L_{2n} + 2F_{2n-1} = M_n L_n + 5F^2{}_n - L^2{}_n$$

Note that Using (9) above , we have $L_{2n} + 2F_{2n-1} = \frac{5F^2{}_n + L^2{}_n}{2} + 2F^2{}_n + 2F^2{}_{n-1} = \frac{9F^2{}_n + L^2{}_n + 4F^2{}_{n-1}}{2}$.

Now Observe that

$$\frac{9F^2{}_n + L^2{}_n + 4F^2{}_{n-1}}{2} = \frac{5F^2{}_n + L^2{}_n}{2} + 2F^2{}_n + 2F^2{}_{n-1}$$
$$= L_{2n} + 2F_{2n-1} = F_{2n} + 2F_{2n-1} + 2F_{2n-1} = F_n L_n + 4F_{2n-1}$$

$$= F_n(F_n + 2F_{n-1}) + 4F^2{}_{n-1} + 4F^2{}_n = 5F^2{}_n + 4F^2{}_{n-1} + 2F_{n-1}F_n =$$
$$= (F^2{}_n + 8F^2{}_{n-1} + 6F_{n-1}F_n) + 5F^2{}_n - (F^2{}_n + 4F_{n-1}F_n + 4F^2{}_{n-1})$$
$$= (F_n + 4F_{n-1})(F_n + 2F_{n-1}) + 5F^2{}_n - (F_n + 2F_{n-1})^2$$
$$= M_n L_n + 5F^2{}_n - L^2{}_n$$

Theorem 3. $M_{2n} = M_n L_n + 5F^2{}_n - L^2{}_n$

Note that

$$M_n L_n + 5F^2{}_n - L^2{}_n = M_n L_n - L^2{}_n + 5F^2{}_n$$
$$= M_n L_n - (F_n + 2F_{n-1})^2 + 5F^2{}_n$$
$$= M_n L_n - (F^2{}_n + 4F_{n-1}F_n + 4F^2{}_{n-1}) + 5F^2{}_n$$

$$= M_n L_n - (F_n + F_{n-1})(F_n + 4F_{n-1}) + F_n F_{n-1} + 5F^2{}_n$$

$$= M_n L_n - F_{n+1} M_n + F_n F_{n-1} + 5F^2_n$$
$$= M_n (L_n - F_{n+1}) + F_n (F_{n-1} + 5F_n)$$
$$= M_n F_{n-1} + F_n M_{n+1}$$
$$= M_{n+n} = M_{2n}, \text{ by } (10) \text{ above}$$

Theorem 4.

$M_{2n} = M_{2n} = M_n L_n + 5F^2_n - L^2_n = M_n L_n + + 4(-1)^{n+1}$, using (8) above. Hence, the theorem follows.

Theorem 5.

(a) If M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) If M_n is divisible by 3, then $M^3_{n+1} - M^3_{n-1}$ is divisible by 9.

Proof: Note that: Using $M_{n+1} = (M_n + M_{n-1})$, we have:

(a) $M^2_{n+1} - M^2_{n-1}$
$$= (M_{n+1} - M_{n-1})(M_{n+1} + M_{n-1}) = M_n (M_n + M_{n-1} + M_{n-1}) = M^2_n + 2M_n M_{n-1}.$$

Now it is easy to see that if M_n is divisible by 2, then $M^2_{n+1} - M^2_{n-1}$ is divisible by 4

(b) $M^3_{n+1} - M^3_{n-1} = (M_{n+1} - M_{n-1})(M^2_{n+1} + M_n M_{n-1} + M^2_{n-1})$
$$= M_n (M^2_{n+1} + M_{n+1} M_{n-1} + M^2_{n-1})$$
$$= M_n ((M_n + M_{n-1})^2 + M_{n-1}(M_n + M_{n-1}) + M^2_{n-1})$$
$$= M_n (M^2_n + 3M_n M_{n-1} + 3M^2_{n-1})$$
$$= M^3_n + 3M^2_n M_{n-1} + 3M_n M^3_{n-1}$$

Hence M_n is divisible by 3 $\Rightarrow$ $M^3_{n+1} - M^3_{n-1}$ is divisible by 9.

Noble Dedication: **Noble Dedication:** We would like to dedicate this paper to Mr. Walden Lambright and Mr. Lemma Bashar who passed away on March, 16, 2016 and December 16, 2014 respectively. Mr. Walden Lambright is the father of Dr. Jonathan Lambright and Mr. Lemma Bashar is the father of Dr. Mulatu Lemma. We thank our fathers for their great support as we are here where we are today because of their unconditional support. We always miss them. Let them rest in peace.

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