*The F*ascinating Mathematical Beauty Of The *Fibonacci* Numbers

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Abstract: The Fibonacci numbers are sequences of numbers of the form: **0,1,1,2,3,5,8,13,...** Among numerical sequences, the Fibonacci numbers F_n have achieved a kind of celebrity status. These numbers are famous for possessing wonderful and amazing properties. Mathematicians have been fascinated for centuries by the properties and patterns of Fibonacci numbers. In mathematical terms, it is defined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_1 = F_2 = 1 \text{ and } F_0 = 0$$

The first number of the sequence is 0, the second number is 1, and each subsequent number is equal to the sum of the previous two numbers of the sequence itself. That is, after two starting values, each number is the sum of the two preceding numbers. In this paper, we give excellent summary of basic properties of Fibonacci numbers as well as and its patterns. This is a paper which is very helpful for quick reference on Fibonacci numbers.

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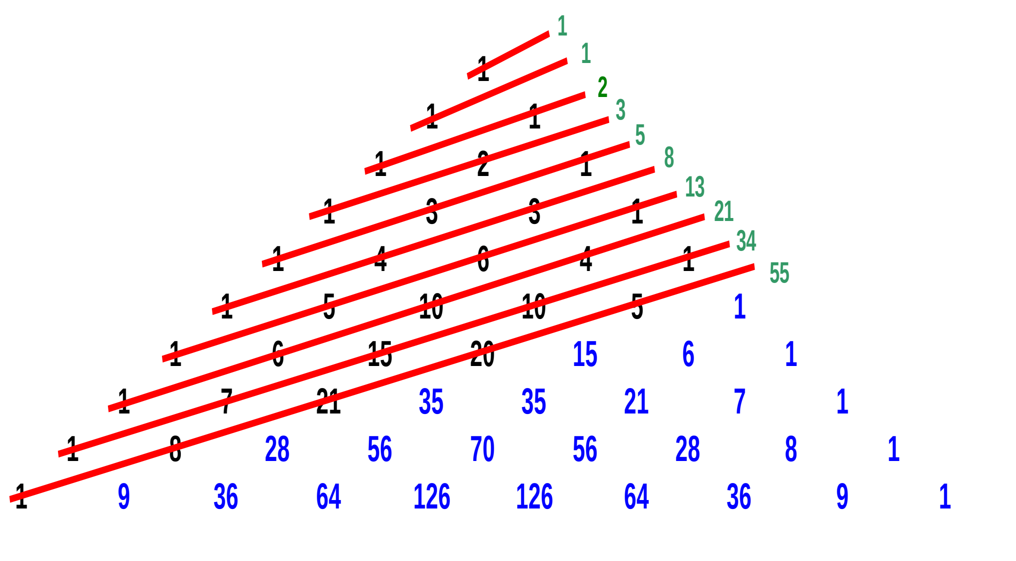
Key Words: Fibonacci numbers, Fibonacci sequences, Pascal's triangle, and Golden ratio.

1. <u>Introduction and Background</u>. The Fibonacci numbers are a sequence of numbers named after Leonardo of Pisa, known as Fibonacci[2]. Fibonacci's 1202 book *Liber Abaci* introduced the sequence to Western European mathematics, although the sequence had been previously described by Indian mathematics [2]. Here are the First 20 Fibonacci numbers.

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765

The Fibonacci numbers appear in an amazingly variety of creations, both natural and people made. These numbers have very interesting properties, and keep popping up in many places in nature and art [1] and [7].

The Fibonacci sequence also makes its appearance in many different ways within mathematics**.** The Fibonacci numbers are studied as part of number theory and have applications in the counting of mathematical objects such as sets, permutations and sequences and to computer science. For example, it appears on the famous Pascal’s triangle as sums of oblique diagonals as shown below.



The Fibonacci numbers and the Fibonacci sequence are prime examples of "how mathematics is connected to seemingly unrelated things." Even though these numbers were introduced in 1202 in Fibonacci's book *Liber abaci*, they remain fascinating to mathematicians still today [2] having amazing mathematical wealth to investigate.

2. The Main Results:

Theorem 1. The sum of the squares of the first n Fibonacci numbers is $F_n F_{n+1}$. That is if each F_i (i≥ 1) are Fibonacci numbers then

$$\sum_{k=0}^{n} F_k^{\ 2} = F_n F_{n+1}$$

Proof: Note that $F_n^{\ 2} = F_n F_n = F_n(F_{n+1} - F_{n-1}) = F_n F_{n+1} - F_n F_{n-1})$ for n≥ 2. Hence, we have

$$\sum_{k=0}^{n} F_k^{\ 2} = F_1^{\ 2} + F_2^{\ 2} + F_3^{\ 2} + \ldots F_n^{\ 2} = F_1^{\ 2} + (\ F_2 F_3 - F_2 F_1) + (F_3 F_4 - F_3 F_2)$$
$$+ (F_4 F_5 - F_4 F_3) + \ldots + (F_{n-1} F_n - F_{n-1} F_{n-2}) + (F_n F_{n+1} - F_n F_{n-1}) = F_1^{\ 2} - F_2 F_1 + F_n F_{n+1} = F_n F_{n+1}$$

Theorem 2. If each F_i (i≥ 1) are Fibonacci numbers, then

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

Proof: We prove using induction on n.

(1) The formula holds for n=1 as F_1 = 1= 2-1 = $F_3 - 1$

(2) Assume the formula is true for n= m. That is

$$\sum_{k=1}^{m} F_k = F_{m+2} - 1$$

(3) Prove that the formula hold for n = m+1. Note that

$$\sum_{k=1}^{m+1} F_k = \sum_{k=1}^{m} F_k + F_{m+1} = F_{m+2} -1+ F_{m+1} = F_{m+3} -1.$$

Hence by mathematical induction, the theorem is proved.

Theorem 3. **T**he sum of the first n Fibonacci numbers with odd indices is F_{2n+2}. That is if each F_i (i≥ 1) are Fibonacci numbers, then

$$\sum_{k=1}^{n} F_{2k+1} = F_{2n+2}\ .$$

Proof: We prove using induction on n.

(1) The formula holds for n=0 as F_1 = 1= F_2

(2) Assume the formula is true for n= m. That is

$$\sum_{k=0}^{m} F_{2k+1} = F_{2m+2}$$

(3) Prove that the formula hold for n = m+1. Note that

$$\sum_{k=0}^{m+1} F_{2k+1} = \sum_{k=1}^{m} F_{2k+1} + F_{2m+3} = F_{2m+2} + F_{2m+3} = F_{2m+4} = F_{2(m+1)+2}.$$

Hence by mathematical induction, the theorem is proved.

Corollary 1. The sum of the first n Fibonacci numbers with even indices is $F_{2n+1} - 1$. That is if each F_i (i≥ 1) are Fibonacci numbers, then $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$.

Proof: We use Theorems 2 and 3 to prove the corollary. From Theorem 2, we have.

$$F_1 + F_2 + F_3 + F_4 + \ldots + F_{2n} = F_{2n+2} - 1$$

$$\Rightarrow (F_1 + F_3 + F_5 + F_7 + \ldots + F_{2n-1}) + (F_2 + F_4 + F_6 + \ldots F_{2n}) = F_{2n+2} - 1.$$

Since $F_1 + F_3 + F_5 + F_7 + \ldots + F_{2n-1} = F_{2n}$ by Theorem 3, it follows that

$$F_{2n} + F_2 + F_4 + F_6 + \ldots F_{2n} = F_{2n+2} - 1$$

$$\Rightarrow F_2 + F_4 + F_6 + \ldots F_{2n} = F_{2n+2} - 1 - F_{2n} = F_{2n+1} + F_{2n} - F_{2n} - 1 = F_{2n+1} - 1$$

$\Rightarrow \sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$. Hence the theorem follows.

Theorem 4: $F^2{}_{n+3} = 2F^2{}_{n+2} + 2F^2{}_{n+1} - F^2{}_n$

Proof: (i) $F^2{}_{n+3} = \left(F_{n+2} + F_{n+1}\right)^2 = F^2{}_{n+2} + 2F_{n+2}F_{n+1} + F^2{}_{n+1}$

(ii) $F^2{}_{n+2} = \left(F_{n+1} + F_n\right)^2 = F^2{}_{n+1} + 2F_{n+1}F_n + F^2{}_n$

$$\Rightarrow F^2_{n+1} = F^2_{n+2} - 2F_{n+1}F_n - F^2_n$$

Now by (i) and (ii) we have

$$\begin{aligned} F^2_{n+3} &= 2F^2_{n+2} + 2F_{n+2}F_{n+1} - 2F_{n+1}F_n - F^2_n \\ &= 2F^2_{n+2} + 2F_{n+1}(F_{n+2} - F_n) - F^2_n \\ &= 2F^2_{n+2} + 2F_{n+1}(F_{n+1}) - F^2_n \\ &= 2F^2_{n+2} + 2F^2_{n+1} - F^2_n \end{aligned}$$

Theorem 5. F_m *and* F_n be any two Fibonacci Numbers. Then we have:

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

Proof: We use induction on n for fixed $m \geq 2$.

(1) When n = 1, the formula is true as

$$F_{m+1} = F_{m-1} + F_m = F_{m-1}F_1 + F_mF_2 .$$

(2). Assume the formula is true for n =1,2,3…k-1, k.

(3) Verify the formula for n= k+1.

Note that

$$\begin{aligned} &(a) \quad F_{m+k} = F_{m-1}F_k + F_mF_{k+1} \\ &(b) \quad F_{m+k-1} = F_{m-1}F_{k-1} + F_mF_k \end{aligned}$$

The addition of the two equations (a) and (b) gives us

$$F_{m+k} + F_{m+(k-1)} = F_{m-1}(F_k + F_{k-1}) + F_m(F_{k+1} + F_k)$$
$$\Rightarrow F_{m+(k+1)} = F_{m-1}F_{k+1} + F_mF_{k+2}$$

Corollary 2. $F_{2n} = F^2_{n+1} - F^2_{n-1}$

Proof: By using Theorem 5 , we have:

$$F_{2n} = F_{n+n} = F_{n-1}F_n + F_nF_{n+1}$$
$$= F_n\left(F_{n-1} + F_{n+1}\right)$$
$$= (F_{n+1} - F_{n-1})\left(F_{n-1} + F_{n+1}\right)$$
$$= F^2{}_{n+1} - F^2{}_{n-1}$$

We state the following important theorem without proof and use it.

Theorem 6. [1]. The greatest common divisor of two Fibonacci numbers F_m and F_n is itself a Fibonacci number and gcd($F_m, F_m) = F_{\gcd(m,n)}$.

T**heorem 7:** Consecutive Fibonacci numbers are relatively prime.

Proof: Consider the two consecutive Fibonacci number F_n and $F_{n+1.}$ Then by Theorem 5, we have gcd (F_n, F_{n+1})= $F_{\gcd(n,n+1)}$ = F_1 =1. Hence, F_n and F_{n+1} are relatively prime.

Theorem 8. If $n \geq m \geq 3$, then F_n is divisible by F_m $\Leftrightarrow$ n is divisible by m.

Proof: (1) Assume is F_n divisible by $F_{m.}$ Then we have

gcd(F_m, F_n)= F_m. By Theorem 5, we also have gcd(F_m, F_n) = $F_{\gcd(m,n)}$. So, we have $F_{\gcd(m,n)}$ = F_m. This implies that gcd (m, n) = m and hence n is divisible by m.

(2) Assume n is divisible by m. Then we have gcd (m, n) = m. This implies that $F_{\gcd(m,n)}$ = F_m and by Theorem 5 it follows that gcd (F_m, F_n)= F_m. Thus,Fn is divisible by $F_{m.}$

Theorem 9. If m≥ 1, n≥ 1, then F_{mn} is divisible by F_m

Proof: We use induction on n for fixed m≥ 1.

(1) When n = 1, the formula is true as F_m is divisible by itself.

(2) Assume the formula is true for n =1,2,3…k-1,k.

(3) Verify the formula for n= k+1.

Note that using Theorem 2.8, we have $F_{m(k+1)} = F_{mk+m} = F_{m(k-1)}\ F_m + F_{mk}F_{m+1}$ and F_{mk} is divisible by $F_m \Rightarrow F_{m(k+1)}$ is also divisible by F_m. Hence the theorem is follows by induction.

Test For Fibonacci number: We state the following theorem with out proof and use it. The theorem helps us on how to identify Fibonacci numbers.

Theorem 10. A positive integer n is a Fibonacci number $\Leftrightarrow\ 5n^2 \pm 4$ is a perfect square [2 page 75]

Example 6. If $\phi = \frac{1+\sqrt{5}}{2}$, show that $\frac{1}{\sqrt{5}}\left(\phi^2 - (1-\phi)^2\right)$ is a Fibonacci number.

Solution: Note that $\frac{1}{\sqrt{5}}\left(\phi^2 - (1-\phi)^2\right)$=1 and 5+4= 9 which is a perfect square.

Hence, $\frac{1}{\sqrt{5}}\left(\phi^2 - (1-\phi)^2\right)$ is a Fibonacci number.

Example 7. Verify that n = 13 is a Fibonacci number.

Solution: We have, $5n^2 \pm 4$ = 5(169) ± 4 = 845 ± 4 and 845-4 = 841 = 29^2 Hence, 13 is A Fibonacci number.

The Golden Ratio.

In mathematics and the arts, two quantities are in the golden ratio if the ratio between the sum of those quantities and the larger one is the same as the ratio between the larger one and the smaller. The golden ratio is an irrational mathematical constant. It is denoted by ϕ . This value is obtained by equating the ratio between the sum of the successive terms and the larger one to the ratio of successive terms in the Fibonacci sequence as shown $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$ That is if m and n are two successive terms in the Fibonacci sequence, we have $\frac{n}{m} \cong \frac{m+n}{n}$

Theorem 11. The Golden Ratios of the Fibonacci Sequence seem to be tending to a limit equal to $\frac{1+\sqrt{5}}{2}$.

Proof: Let m, n , and m + n be successive terms of the sequence.

Then we have $\frac{n}{m} \cong \frac{m+n}{n}$

$$\Rightarrow \frac{n}{m} \cong 1 + \frac{m}{n}$$

Defining ϕ to be the limit of $\frac{n}{m}$, we have $\phi = 1 + \frac{1}{\phi}$.

$\Rightarrow \phi^2 = \phi + 1 \Rightarrow \phi^2 - \phi - 1 = 0$. Using Quadratic Formula, we get $\phi = \frac{1+\sqrt{5}}{2}$.

Binet's Formula. We state the following Binet's Formula for F_n found in [2] and [6] without proof as Lemma 1 and use it.

Lemma 1: $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\phi^n - (1-\phi)^n\right)$

Theorem 12. $\sum_{k=0}^{n} \binom{n}{k} F_k = F_{2n}$

Proof: By Lemma 1 and Binomial Theorem, we have

$$\sum_{k=0}^{n} \binom{n}{k} F_k$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} \left(\phi^k + (1-\phi)^k\right)$$

$$= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} \left((\phi)^k - \left(1-\phi\right)^k\right)$$

$$= \frac{1}{\sqrt{5}}((1+\phi)^n - (1+(1-\phi))^n) = \frac{1}{\sqrt{5}}(\phi^{2n} - (1-\phi)^{2n}) = F_{2n}$$

Fibonacci primes

A Fibonacci prime is a Fibonacci number that is prime. The first few Fibonacci primes are

2, 3, 5, 13, 89, 233, 1597,...

Note that every Fibonacci number F_n is prime only if n prime (n=p) except n=4. The sufficient condition is false, however

Open Question. *Is there an infinite number of Fibonacci primes?*

Dedication: I would like to dedicate this paper to Dr. Chellu Chetty, Associate VP for Research Sponsored Programs, for his great encouragement and fruitful supports which have helped me bring distinction and honor to my profession by excelling in teaching, research, grant ship, and mentoring.

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