A NUMERICAL METHOD OF SOLVING VOLTERRA INTEGRAL EQUATION

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Abstract

A numerical method is presented in this paper to solve linear Volterra integral equations of the second kind. In this proposed method, orthogonal Legendre polynomials are employed to approximate a solution for an unknown function in the Volterra integral equation and convert the equation to system of linear algebraic equations. The accuracy and efficiency of this method are illustrated through some numerical examples.

Keywords: Volterra integral equation, Legendre polynomial, operational matrix, function approximation

1. Introduction

This paper presents a method of solving Linear Volterra integral equations of the second kind have the form

$$y(x) - \int_a^x k(x,t)y(t)dt = f(x), -\infty < a \le x \le b < \infty \tag{1}$$

using Legendre polynomials. In solving the integral equations with given kernel k(x, t) and the function f(x), the problem is typically to find the unknown function y(x).

Legendre polynomial is an important orthogonal polynomial with interval of orthogonality between -1 and 1, and also is considered as the eigenfunctions of singular Sturm-Liouville [1]. In mathematics, Legendre polynomials are solutions to Legendre's differential equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n(x)\right] + \lambda P_n(x) = 0 \tag{2}$$

where the eigenvalue λ equals n(n+1). The recurrence relation of Legendre polynomials is

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x), n \ge 1 \tag{3}$$

and Rodrigues' formula is

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(\left[x^2 - 1\right]^n\right) \tag{4}$$

Because of its orthogonal properties, Legendre polynomials have been used for solving other integral equations such as Fredholm integral equations [2]. This paper employs Legendre polynomials to expand the unknown function y(x) in Eq. (1) as Legendre polynomials series with unknown coefficients. The unknown coefficients are then determined based on the properties of the Legendre polynomials and some operational matrices. The detailed approach is demonstrated in following sections and validated through a numerical example.

2. Function approximation

A function f(x) defined on [-1, 1] can be expanded by Legendre polynomials series as

$$f(x) = \sum_{i=0}^{\infty} c_i P_i(x) \tag{5}$$

If the infinite series in Eq. (5) is truncated, then we have

$$f(x) \approx \sum_{i=0}^{N} c_i P_i(x) = P^T(x)C \tag{6}$$

where C and P(x) are (N + 1) × 1 vectors given by

$$C = [c_0, c_1, c_2, \;\; , c_N]^{\mathrm{T}} \text{ and } P(x) = [P_0(x), P_1(x), P_2(x), \;\; , P_N(x)]^{\mathrm{T}} \tag{7}$$

where the coefficients c_i are determined by Scanlon [3]

$$c_i = \frac{2i+1}{2}\int_{-1}^{1} f(x)P_i(x)dx, \text{ i = 0, 1, …, N} \tag{8}$$

The kernel k(x, t) is estimated as

$$k(x,t) \approx P^T(x)KP(t) \tag{9}$$

where K is a (N+1)×(N+1) matrix, with

$$K_{ij} = \left(P_i(x), k(x,t), P_j(t)\right) = \left(\frac{2i+1}{2}\right)\left(\frac{2j+1}{2}\right)\int_{-1}^{1}\int_{-1}^{1} P_i(x)k(x,t)P_j(t)dxdt \tag{10}$$

where (.,.) denotes the inner product.

3. Solving Volterra integral equations

In this section, the Volterra integral equation of the second kind (eq. (1)) is solved by using Legendre polynomials $P_i(x)$. As demonstrated in above sections, the unknown function y(x), and the kernel k(x, t) are firstly approximated as

$$y(x) = P^T(x)Y \text{ and } k(x, t) = P^T(x)KP(t) \tag{11}$$

The integral $\int_{-1}^{x} k(x,t)y(t)dt$ is approximated as

$$\int_{-1}^{x} k(x,t)y(t)dt \approx \int_{-1}^{x} P^T(x)KP(t)P^T(t)Ydt = P^T(x)K\left(\int_{-1}^{x} P(t)P^T(t)dt\right)Y \tag{12}$$

Here we have to simplify $\int_{-1}^{x} P(t)P^T(t)dt$.

Here we assume a (N+1) × (N+1) square matrix Z(x) whose elements z_{ij} are, which can be easily calculated based on a given x.

$$z_{ij} = \int_{-1}^{x} P_i(t)P_j(t)dt \quad (13)$$

Substituting eq. (11 – 13) into eq. (1) and have

$$y(x) - \int_{-1}^{x} k(x,t)y(t)dt = P^T(x)Y - P^T(x)KZ(x)Y = f(x) \quad (14)$$

To find the unknown coefficient vector Y, select N + 1 collocation points x_i in the interval [-1, 1] that

$$x_i = -1 + \frac{2i}{N}, \ i = 0, 1, \ldots, N \quad (15)$$

Eq. (14) is therefore reduced to a system of N + 1 linear algebraic equations

$$P^T(x_i)Y - P^T(x_i)KZ(x_i)Y = f(x_i) \quad (16)$$

where $P^T(x_i)$ and $f(x_i)$ are N + 1 vectors, K and $Z(x_i)$ are (N+1) × (N+1) square matrices, which can be calculated from above equations based on specified x_i. The vector Y can be easily solved as

$$Y = \left(P^T(x_i) - P^T(x_i)KZ(x_i)\right)^{-1} f(x_i) \quad (17)$$

The unknown function y(x) is then approximated using the Legendre polynomials with coefficients Y.

4. Numerical examples

A example of the Volterra integral equation (eq. (1)) is solved to illustrate the accuracy of presented method. These problems have been previously solved by Mestrovic and Maleknejad [4]. All calculations are performed using Maple 11 and Matlab 7.0; the detailed steps are:

1. Approximate y(x) and k(x, t) with Legendre polynomials and start with eq. (14) (eq. (11, 14)).
2. Determine the order N and select collocation points (eq. (15)).
3. Calculate square matrix K for given kernel k(x, t) (eq. (10).
4. Evaluate vectors $P^T(x_i)$ and $f(x_i)$, and compute square matrix $Z(x_i)$ for every collocation points x_i (eq. (16)).
5. Return to eq. (17) and solve Y as well as y(x).

Numerical results are compared with the exact solutions and plotted in following figures to validate the proposed method.

Example Solve the linear Volterra integral equation of the second kind

$$y(x)-\int_{-1}^{x} e^{-x^2+t^2}\, y(t)dt = 2x-(1-e^{1-x^2})\text{ , } -1 \le x \le 1$$

where k(x, t) = $e^{-x^2+t^2}$ and f(x) = $2x-(1-e^{1-x^2})$ with exact solution y(x) = 2x.

We solve this problem following above steps with N = 10. The obtained results are compared to the exact solution and the absolute errors between the estimated solution and the exact one are plotted in figure 1. From that figure it can be found that the errors are below 1.3×10^{-5}.

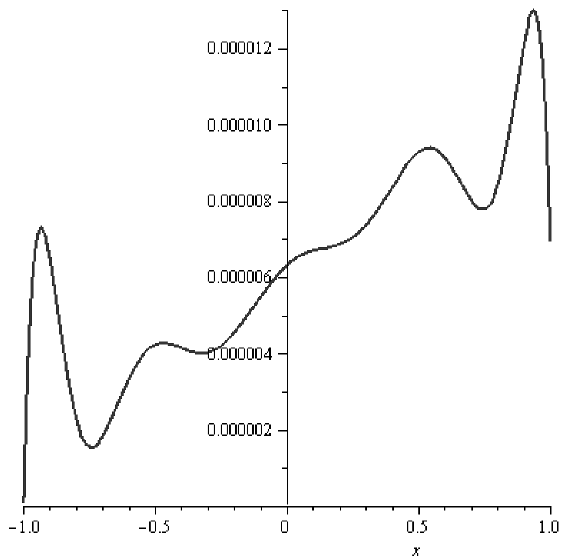

Figure 1. Absolute error in example 1 for N = 10

5. Conclusion

In this paper, the linear Volterra integral equation of the second kind is solved by Legendre polynomials and collocation method. In the presented method, the unknown function y(x) is approximated using Legendre polynomials and the integral equation is converted to a system of algebraic equations. As verified through the numerical example, the proposed method provides good efficiency that in order to acquire enough accuracy, we only need to convert the integral equation to the system of linear equations by order less than ten.

Reference

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