A NUMERICAL METHOD OF SOLVING NONLINEAR DIFFERENTIAL DIFFERENCE EQUATIONS

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Abstract – This paper presents a method of solving the nonlinear differential difference equations (NDDEs) using He's variational iteration method. In this study, typical nonlinear differential difference equations such as the discretized mKdV lattice equation and the Toda lattice equation were solved using the He's method. The computed results were compared to the results obtained by Adomian decomposition method (ADM) and the exact analytical results. A good agreement was achieved through the comparisons and the present method for NDDEs is found to be accurate and simple.

Keywords: He's variational iteration method; nonlinear differential difference equation; discretized nonlinear equation

Introduction

The objective of this paper is to apply He's variational iteration method for solving the NDDEs. This method was presented by Chinese mathematician Ji-Huan He as a modification of a general Lagrange multiplier method [1]. Particularly, in this paper, the He's variational iteration method is applied to solve the discretized mKdV lattice equation [3]:

$$\frac{du_n}{dt} = \left(\alpha - u^2{}_n\right)\left(u_{n+1} - u_{n-1}\right) \tag{1}$$

and the Toda lattice equation [4]:

$$\begin{aligned} \frac{du_n}{dt} &= u_n\left(v_n - v_{n-1}\right) \\ \frac{dv_n}{dt} &= v_n\left(u_{n+1} - u_n\right) \end{aligned} \tag{2}$$

where the subscript n in Eqns. (1) and (2) represents the nth lattice.

Applications of He's Method

In this section, two NDDEs (Eqns. (1) and (2)) were solved using the He's method and the obtained solutions were compared with both the exact analytical solutions and the solutions yielded from ADM method [2] to illustrate the accuracy and advantages of the He's method.

Example 1. We consider the Eqn. (1) with the parameter $\alpha = 1$ and initial condition

$$u_0(n, t) = \tanh(k)\tanh(kn) \tag{6}$$

By comparing the Eqn. (1) with Eqn. (4) the correctional function can be written as

$$u_{p+1}(n,t) = u_p(n,t) + \int_0^t \lambda \left(\frac{du_p(n,x)}{dx} - \left(1 - u^2{}_p(n,x)\right)\left(u_p(n+1,x) - u_p(n-1,x)\right) \right) dx \tag{7}$$

Eqn. (7) can be simplified as

$$u_{p+1}(n,t) = u_p(n,t) + \lambda u_p(n,t) - \int_0^t \lambda^{(1)} u_p(n,x)dx - \int_0^t \lambda u_p(n+1,x)dx + \int_0^t \lambda u_p(n-1,x)dx + \int_0^t \lambda u^2{}_p(n,x)\left(u_p(n+1,x) - u_p(n-1,x)\right)dx \tag{8}$$

By taking variation on both sides of Eqn. (8) with respect to u_p, one can have

$$\delta u_{p+1}(n,t) = \delta u_p(n,t) + \lambda \delta u_p(n,t) - \int_0^t \lambda^{(1)} \delta u_p(n,x)dx - \int_0^t \lambda \delta u_p(n+1,x)dx + \int_0^t \lambda \delta u_p(n-1,x)dx + \int_0^t \lambda \delta u^2{}_p(n,x)\left(u_p(n+1,x) - u_p(n-1,x)\right)dx \tag{9}$$

Since $u^2{}_p$ is a nonlinear term and $\delta u^2{}_p$ becomes zero, Eqn. (9) becomes

$$\delta u_{p+1}(n,t) = \delta u_p(n,t) + \lambda \delta u_p(n,t) - \int_0^t \lambda^{(1)} \delta u_p(n,x)dx - \int_0^t \lambda \delta u_p(n+1,x)dx + \int_0^t \lambda \delta u_p(n-1,x)dx \tag{10}$$

The stationary conditions obtained from Eqn. (10) are

$$1 + \lambda = 0 \tag{11}$$

$$\lambda^1|_{x = t} = 0 \tag{12}$$

From Eqn. (11), the Lagrange multiplier λ is found as –1. Substituting $\lambda = -1$into Eqn. (7) so that the correctional function becomes

$$u_{p+1}(n,t) = u_p(n,t) - \int_0^t \left(\frac{du_p(n,x)}{dx} - \left(1 - u^2{}_p(n,x)\right)\left(u_p(n+1,x) - u_p(n-1,x)\right) \right) dx \tag{13}$$

From Eqns. (13), (6), and (5) the approximate solution to the Eqn. (1) can be determined after running four iterations.

Example 2. We consider the Eqn. (2) with initial conditions

$$u_0(n, t) = -\coth(d)c + c\tanh(dn),\ v_0(n, t) = -\coth(d)c - c\tanh(dn) \tag{14}$$

By comparing Eqns. (3), (4) and (14), the correctional function can be expressed as

$$
\begin{aligned}
u_{p+1}(n,t) &= u_p(n,t) + \int_0^t \lambda_1\left(\frac{d}{dt}u_p(n,x) - u_p(n,x)\big(v_p(n,x) - v_p(n-1,x)\big)\right)dx \\
v_{p+1}(n,t) &= v_p(n,t) + \int_0^t \lambda_2\left(\frac{d}{dt}v_p(n,x) - v_p(n,x)\big(u_p(n+1,x) - u_p(n,x)\big)\right)dx
\end{aligned}
\tag{15}
$$

which can be further simplified as

$$
\begin{aligned}
u_{p+1}(n,t) &= u_p(n,t) + \lambda_1 u_p(n,t) - \int_0^t \lambda_1^1 u_p(n,x)dx - \int_0^t \lambda_1\big(u_p(n,x)\big(v_p(n,x) - v_p(n-1,x)\big)\big)dx \\
v_{p+1}(n,t) &= v_p(n,t) + \lambda_2 v_p(n,t) - \int_0^t \lambda_2^1 v_p(n,x)dx - \int_0^t \lambda_2\big(v_p(n,x)\big(u_p(n+1,x) - u_p(n,x)\big)\big)dx
\end{aligned}
\tag{16}
$$

Applying variance on both sides of the two equations in Eqn. (16) with respect to u_p and v_p separately and we can have

$$
\begin{aligned}
\delta u_{p+1}(n,t) &= \delta u_p(n,t) + \lambda_1 \delta u_p(n,t) - \int_0^t \lambda_1^1 \delta u_p(n,x)dx - \int_0^t \lambda_1\big(\delta u_p(n,x)\big(\delta v_p(n,x) - \delta v_p(n-1,x)\big)\big)dx \\
\delta v_{p+1}(n,t) &= \delta v_p(n,t) + \delta\lambda_2 v_p(n,t) - \int_0^t \lambda_2^1 \delta v_p(n,x)dx - \int_0^t \lambda_2\big(\delta v_p(n,x)\big(\delta u_p(n+1,x) - \delta u_p(n,x)\big)\big)dx
\end{aligned}
\tag{17}
$$

The stationary conditions obtained from above equations

$$1 + \lambda_1 = 0 \tag{18}$$

$$\lambda_1^{1}|_{x = t} = 0 \tag{19}$$

$$1 + \lambda_2 = 0 \tag{20}$$

$$\lambda_2^{1}|_{x = t} = 0 \tag{21}$$

Solving Eqns. (18) to (21) one can obtain

$$\lambda_1 = \lambda_2 = -1 \tag{22}$$

Substituting λ_1 and λ_2 into Eqn. (16) and we have

$$
\begin{aligned}
u_{p+1}(n,t) &= u_p(n,t) - \int_0^t \left(\frac{d}{dt}u_p(n,x) - u_p(n,x)\big(v_p(n,x) - v_p(n-1,x)\big)\right)dx \\
v_{p+1}(n,t) &= v_p(n,t) - \int_0^t \left(\frac{d}{dt}v_p(n,x) - v_p(n,x)\big(u_p(n+1,x) - u_p(n,x)\big)\right)dx
\end{aligned}
\tag{23}
$$

Similarly, from Eqns. (15), (23), and (5) the approximate solution to Eqn. (2) can be calculated after running four iterations.

Comparison of Approximate Solutions

After solving examples 1 and 2, the numerical solutions (u_{appr}) were compared to the approximate solutions obtained using ADM and the exact solutions in order to verify the accuracy of the proposed method. The exact solutions for example 1 is a kink-type soliton solution as presented in [5]:

$$u = \tanh(k)\tanh(kn + 2\tanh(k)t) \quad (24)$$

Tables 1 and 2 compare the solution for example 1 and the results are visualized in Fig. 1.

For example 2, the obtained four-term approximate solutions u_{appr} and v_{appr} were compared with the exact solutions, which are

$$u_0(n, t) = -\coth(d)c + c\tanh(dn + ct),\ v_0(n, t) = -\coth(d)c - c\tanh(dn + ct) \quad (25)$$

and the results are listed in tables 3 to 6 and plotted in Figs. 2 and 3.

These tables and figures show the numerical approximate solution obtained using He's method is of high accuracy. As we know, the more terms added to the approximate solution, the more accurate it will be. Although we only ran 4 iterations, the solution achieves a high level of accuracy.

Table 1 (For constant k = 0.1, and time t = 0.5)

n	ADM	He's method	Exact solution	Relative error	Absolute error
-25	-0.098041971660	-0.098042053466	-0.098041971669	8.3440E-7	8.3431E-7
-15	-0.088248372980	-0.088248609374	-0.088248373764	2.67874E-6	2.66985E-6
-5	-0.037897066100	-0.037895459388	-0.037897060908	4.239674E-5	4.225975E-5
0	0.009900946992	0.009900819170	0.009900946469	1.291007E-5	1.285727E-5
5	0.053503102820	0.053501551977	0.053503102830	2.898604E-5	2.898622E-5
15	0.091855587327	0.091856115038	0.091855874015	5.74500E-6	2.62393E-6
25	0.098573655420	0.098573731584	0.098573655380	7.7266E-7	7.7307E-7

Table 2 (For constant k = 0.1, and time t = 1.5)

n	ADM	He's method	Exact solution	Relative error	Absolute error
-25	-0.09725516662	-0.097262239448	-0.097255114012	7.272444E-5	7.326541E-5
-15	-0.08311834180	-0.083137056072	-0.083118937378	2.2515213E-4	2.1798515E-4
-5	-0.01977150813	-0.019637206900	-0.019767387203	6.79266491E-3	6.58560996E-3
0	0.02894478018	0.028913718993	0.028943671058	1.07311877E-3	1.03483987E-3
5	0.06613063122	0.066008779302	0.066127678612	1.84259421E-3	1.79802637E-3
15	0.09435553904	0.094375549578	0.094355966563	2.1207592E-4	2.0754401E-4
25	0.09893218337	0.098937897329	0.098932134016	5.775633E-5	5.825522E-5

Note: relative error is calculated by comparing the numerical solutions to the solutions obtained using ADM and absolute error is calculated by comparing them to the exact solution

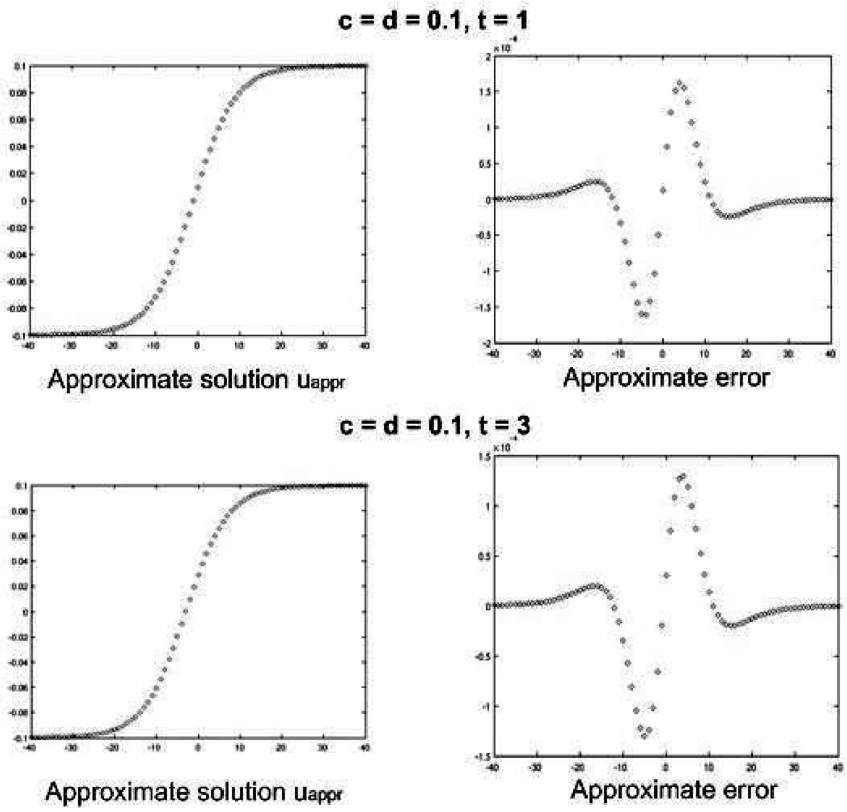

Figure 1. u_{appr} compared with the exact solution when k = 0.1

Table 3 (For results u, constant c = d = 0.1, and time t = 1)

n	He's method	Exact solution	Absolute error
-25	-1.101698601961	-1.101698598995	2.6925E-7
-15	-1.091866275770	-1.091866278046	2.08451E-7
-5	-1.041325978913	-1.041326009451	2.93264E-6
0	-0.993364443034	-0.993364313763	1.30135E-5
5	-0.949626135910	-0.949626156526	2.17095E-6
15	-0.911164254386	-0.911164257785	3.72962E-7
25	-0.904428375739	-0.904428373005	3.02244E-7

Table 4 (For results u, constant c = d = 0.1, and time t = 3)

n	He's method	Exact solution	Absolute error
-25	-1.100906183816	-1.100905426229	6.8815E-5
-15	-1.086695607968	-1.086696573927	8.88894E-5
-5	-1.023064082946	-1.023068645248	4.45943E-4
0	-0.974230377885	-0.974199851980	3.133434E-3
5	-0.936920483002	-0.936927436199	7.42128E-4
15	-0.908649971737	-0.908650511941	5.94512E-5
25	-0.904068592750	-0.904067961205	6.98559E-5

Table 5 (For results v, constant c = d = 0.1, and time t = 1)

n	He's method	Exact solution	Absolute error
-25	-0.904963624581	-0.904963627456	3.17646E-7
-15	-0.914795953280	-0.914795948405	5.32833E-7
-5	-0.965336227329	-0.965336217000	1.07002E-6
0	-1.013297783854	-1.013297912688	1.27143E-5
5	-1.057036110193	-1.057036069925	3.80955E-6
15	-1.095497969571	-1.095497968666	8.25877E-8
25	-1.102233850625	-1.102233853446	2.5591E-7

Table 6 (For results v, constant c = d = 0.1, and time t = 3)

n	He's method	Exact solution	Absolute error
-25	-0.905756066349	-0.905756800222	8.10232E-5
-15	-0.919967280167	-0.919965652524	1.76924E-4
-5	-0.983593131668	-0.983593581203	4.57033E-5
0	-1.032432166561	-1.032462374471	2.925812E-3
5	-1.069746349899	-1.069734790252	1.080609E-3
15	-1.098011669416	-1.098011714510	4.1069E-6
25	-1.102593613335	-1.102594265246	5.91251E-5

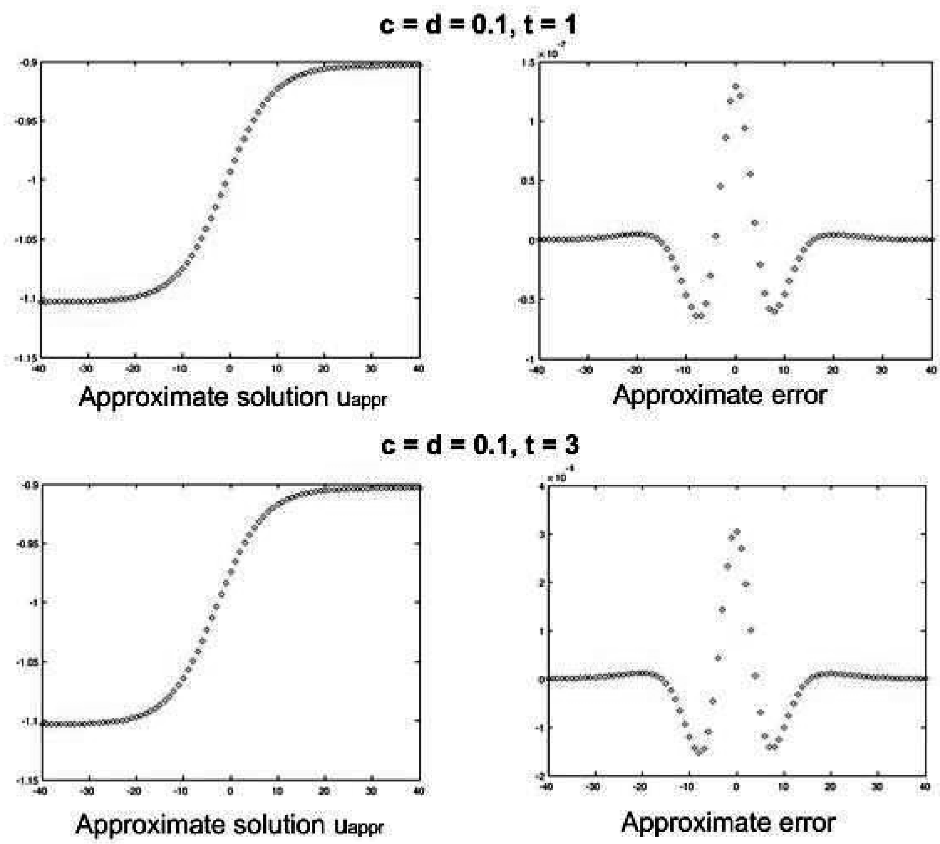

Figure 2. u_{appr} compared with the exact solution when c = d = 0.1

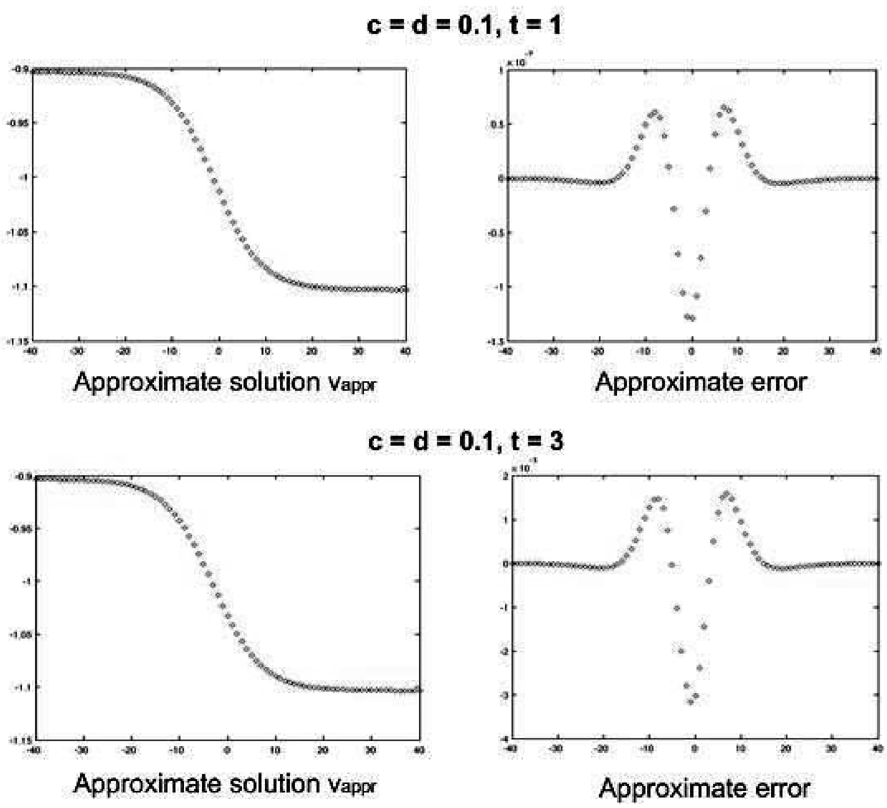

Figure 3. v_{appr} compared with the exact solution when c = d = 0.1

Conclusion

In this paper, He's variational iteration method was applied to find the approximate solutions for the NDDEs, including the discretized mKdV lattice equation and Toda lattice equation. The obtained solutions agreed very well to the exact analytical solutions and the approximate solutions obtained using ADM. Comparing to the extended ADM presented by Wu [2], it can be found that He's method provides two advantages in solving NDDEs. First, He's method involves less number of calculations. In using He's method, each iteration step gives direct approximate solution to the problems. However, in using ADM, each iteration step only gives components to the approximate solution. Therefore, after each step, these components have to be summed up in order to have the approximate solution and check the accuracy. Second, ADM can be directly employed to solve linear equations. Nevertheless, in using ADM to solve nonlinear equations, the Adomian polynomials have to be calculated, which causes the whole problem-solving process more complicated and time consuming. Fortunately, He's method does not require any extra calculations in solving nonlinear equations. It is the same in using He's method to solve either linear or nonlinear equations. The presented examples and comparisons verified the accuracy and efficiency of using He's method to solve the NDDEs.

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