2013 Hawaii University International Conferences
Education & Technology
Math & Engineering Technology
June 10th to June 12th
Ala Moana Hotel, Honolulu, Hawaii

Fibonacci Sequences and Toeplitz Matrices: Research Opportunities for Undergraduates

Dr. Ben Mathes

Colby College

Fibonacci sequences and Toeplitz matrices: research opportunities for undergraduates

Ben Mathes

February 26, 2013

1 Introduction

Most pure mathematicians, upon receiving a doctoral degree, find themselves armed with very specific *state of the art* tools for use in further mathematical investigation. Modular forms, homologies and cohomologies, K-theories and operator algebras are just a sampling of the high powered ammunition that a young mathematician might have in her arsenal. When letters are solicited concerning the depth and quality of research, we often find comments concerning the methods used and the sophistication of the techniques. If we work at an institution that relies on such letters in their promotion process, it is natural to find ourselves skeptical of the value of undergraduate research in pure mathematics.

This paper will document research done recently with undergraduates at Colby College. This experience has converted me from a skeptic to an enthusiastic proponent of undergraduate research activity in pure math. What we find is a learning experience that teaches crucial skills not found in the traditional curriculum. Two such skills include the ability to ask good questions and the art of working on open ended problems.

2 Fibonacci

The Fibonacci sequence is defined by indicating the two beginning "seed" values $f_0 = 0$ and $f_1 = 1$, then continued recursively by $f_{n+1} = f_n + f_{n-1}$. It is one member of a family of recursive sequences that satisfy the same recursive relation, but have different seed values, and that family is denoted $\mathcal{R}(1, 1)$. Folks generally refer to every element of $\mathcal{R}(1, 1)$ as *a Fibonacci sequence*, though the one indicated above retains special status and is *the*

Fibonacci sequence. The point is that, every sequence in $\mathcal{R}(1,1)$ has remarkable properties that were first discovered for *the* Fibonacci sequence, but are now known to be a consequence of the recursion relation, not the seed values. More generally, one denotes by $\mathcal{R}(a,b)$ the set of sequences $\{s_n\}$ that satisfy

$$s_{n+1} = as_n + bs_{n-1},$$

obtaining one member for each seed value pair. Many of the remarkable identities and properties first discovered for the Fibonacci sequence extend easily to this level of generality. These are the generalized Fibonacci sequences. See [5] for an excellent overview of the subject.

Let us consider *the* Fibonacci sequence, defined above and denoted from here on by $\{f_n\}$, and a companion sequence $\{l_n\}$ in $\mathcal{R}(1,1)$, called the *Lucas sequence*, with seed values $l_0 = 2$ and $l_1 = 1$. Perhaps the first mind-blowing properties we see are the profusion of identities these sequences satisfy, some of them almost unbelievable at first. For example, recall that the length of a vector $\vec{v}$ is

$$||\vec{v}||^2 = \sum_{i=1}^{n} v_i^2.$$

If $v_f = (f_1, \ldots, f_n)$ and $v_l = (l_1, \ldots, l_n)$, then

$$||v_f||^2 = f_n f_{n+1} \text{ and } ||v_l||^2 = l_n l_{n+1} - 2.$$

In general, there is a simple formula for $||v_s||^2$ for any sequence in $\mathcal{R}(1,1)$, and this formula easily generalizes to sequences in $\mathcal{R}(a,1)$.

The project that culminated in the paper [6] was motivated by the papers [1], [2], [3], [9], [10], and [11]. These authors were estimating spectral norms (see the appendix for definitions) of various patterns of matrices built using the Fibonacci and Lucas sequences. We were able to obtain the exact values of these norms, and the idea underlying the solution is a pretty one: just as second order recursive sequences lead to beautiful scalar identities, we were able to obtain *matricial identities* that encode literally dozens of scalar identities, and using the matricial identities, we computed the exact values of the norms.

We are concerned with the matrices

$$F = (f_{i-j})_{i,j=o}^{n-1} \text{ and } L = (l_{i-j})_{i,j=o}^{n-1},$$

with f and l denoting the Fibonacci and Lucas sequences, and we refer to F and L as the Fibonacci and Lucas matrices. An $n \times n$ matrix $A = (a_{ij})$ is called *Toeplitz* when there is a sequence $(\alpha_k)_{k=1}^{2n-1}$ such that $a_{ij} = \alpha_{i-j}$ for all i, j. This is a precise way of saying that the matrix A is constant along all its upper left to lower right diagonals. For example, both F and L are the Toeplitz matrix

$$\begin{bmatrix} s_0 & s_1 & \cdots & s_{n-3} & s_{n-2} & s_{n-1} \\ s_{-1} & s_0 & s_1 & \cdots & s_{n-3} & s_{n-2} \\ \vdots & s_{-1} & \ddots & \ddots & & s_{n-3} \\ s_{-n+3} & \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{-n+2} & s_{-n+3} & & \ddots & s_0 & s_1 \\ s_{-n+1} & s_{-n+2} & s_{-n+3} & \cdots & s_{-1} & s_0 \end{bmatrix}$$

with $s_i = f_{k,i}$ and $s_i = l_{k,i}$ (respectively). Note the negative subscripts on the recursive sequences: the fact is, any element of $\mathcal{R}(a, b)$ can be extended backward via the recurrence relation. For example, to compute f_{-1}, simply write $f_{-1} + f_0 = f_1$ and solve.

The unitary matrix

$$U = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

plays a central role in [6]. This is the matrix for which UA reverses the order of the rows of A, and AU reverses the order of the columns of A.

3 Matricial identities

3.1 Even dimensions

The nicest matrix identities arise in the even dimensional case, for reasons that became apparent to us when we later worked on Hankel matrices [7]. The reader might compute the northwest entry of the leftmost matrix to gain an appreciation of the scalar identities encoded in the following. We denote the transpose of a real matrix A by A^*.

Theorem 1 *[6] Let F denote the $n \times n$ Fibonacci matrix with n even. Then*

$$FUF = f_n F.$$

Theorem 2 *[6] Let F denote the $n \times n$ Fibonacci matrix with n even. Then*

$$(F^*F)^2 = f_n^2(F^*F).$$

Theorem 3 *[6] Let F denote the $n \times n$ Fibonacci matrix, L the $n \times n$ Lucas matrix, and assume that n is even. Then*

$$LUL = 5f_nF \text{ and } L^*UL^* = 5f_nF^*.$$

Theorem 4 *[6] Let F denote the $n \times n$ Fibonacci matrix, L the $n \times n$ Lucas matrix, and assume that n is even. Then*

$$L^*L = 5F^*F.$$

Theorem 5 *[6] Assume n is even and F and L denote the $n \times n$ Fibonacci and Lucas matrices. We have*

$$LUF = FUL = f_nL.$$

Given two $n \times n$ matrices A and B, define a multiplication $\circ$ by

$$A \circ B \equiv \frac{1}{f_n} AUB.$$

Let $\mathcal{A}$ denote the two dimensional linear span of F and L. It follows that $\mathcal{A}$ consists of all matrices of the form

$$\begin{bmatrix} s_0 & s_1 & \cdots & s_{n-3} & s_{n-2} & s_{n-1} \\ s_{-1} & s_0 & s_1 & \cdots & s_{n-3} & s_{n-2} \\ \vdots & s_{-1} & \ddots & \ddots & & s_{n-3} \\ s_{-n+3} & \vdots & \ddots & \ddots & \ddots & \vdots \\ s_{-n+2} & s_{-n+3} & & \ddots & s_0 & s_1 \\ s_{-n+1} & s_{-n+2} & s_{-n+3} & \cdots & s_{-1} & s_0 \end{bmatrix}$$

with $\{s_i\} \in \mathcal{R}(1,1)$. If, instead of usual matrix multiplication, we endow upon $\mathcal{A}$ the newly defined multiplication $\circ$, then we have that

$$F \circ (aF + bL) = (aF + bL) \circ F = (aF + bL),$$

for all $a, b \in \mathbb{C}$, and

$$(aF + bL) \circ (cF + dL) = (ac + 5bd)F + (ad + bc)L,$$

which proves

Theorem 6 *[6] The space $\mathcal{A}$ with the multiplication $\circ$ is a commutative algebra with involutive element $\frac{1}{\sqrt{5}}L$ and unit F, isomorphic to the set of all complex matrices of the form*

$$\begin{pmatrix} a & 5b \\ b & a \end{pmatrix}.$$

The fact that F plays the role of the unit element of $\mathcal{A}$ is related to the method we use to compute the spectral norm of F. Recall that the spectral norm of a matrix A, which we denote $||A||$, is the largest singular value of $(A^*A)^{\frac{1}{2}}$ (see the appendix). By theorem 2, we have that $\frac{1}{f_n^2}F^*F$ is self-adjoint and

$$(\frac{1}{f_n^2}(F^*F))^2 = \frac{1}{f_n^4}f_n^2(F^*F) = \frac{1}{f_n^2}(F^*F).$$

It follows that $\frac{1}{f_n^2}F^*F$ is an orthogonal projection, which has spectral norm one, and hence

$$||F||^2 = ||F^*F|| = f_n^2||\frac{1}{f_n^2}F^*F|| = f_n^2.$$

This proves the following theorem.

Theorem 7 *[6] Let F denote the $n \times n$ Fibonacci matrix with n even. Then*

$$||F|| = f_n.$$

In a similar vein, one proves that

Theorem 8 *[6] If L is the $n \times n$ Lucas matrix with n even, then*

$$||L|| = \sqrt{5}||F|| = \sqrt{5}f_n.$$

3.2 Odd dimensions.

It is not true that $FUF = f_nF$ when n is odd. When n is even, we also have the identity

$$FUFUF = f_nFUF = f_n^2F,$$

and, as it happens, this identity almost holds for odd n too.

Theorem 9 *[6] Assume F is a Fibonacci matrix. Then F satisfies*

$$FUFUF = \alpha F.$$

If F is $n \times n$ with n even, then $\alpha = f_n^2$, otherwise $\alpha = f_n^2 - 1$.

Theorem 10 *[6] Assume F is an $n \times n$ Fibonacci matrix. Then we have*

$$(F^*F)^3 = \alpha^2 F^*F = \beta F^*F.$$

If n is even, then $\beta = f_n^4$, otherwise $\beta = (f_n^2 - 1)^2$.

Proof. Using the fact that $UFU = F^*$ and $UF^*U = F$ we have

$$(F^*F)^3 = F^*(F)F^*F(F^*)F = F^*UF^*UF^*FUFUF = \alpha^2 F^*F,$$

with α as in the previous theorem.

□

Theorem 11 *[6] Assume F is a Fibonacci matrix. Then*

$$||F|| = \sqrt{\alpha} = \sqrt{\sum_{i=0}^{n-1} f_i f_{i+1}},$$

with α as in Theorem 9. In case F is $n \times n$ with n odd, this expression simplifies to

$$||F|| = \sqrt{f_n^2 - 1}.$$

Proof. The fact that $(F^*F)^3$ is $\alpha^2 F^*F$ implies that $\frac{1}{\alpha}F^*F$ is an orthogonal projection, so

$$||\frac{1}{\alpha}F^*F|| = 1.$$

It follows that

$$||F||^2 = ||F^*F|| = \alpha||\frac{1}{\alpha}F^*F|| = \alpha.$$

The identity

$$\alpha = \sum_{i=0}^{n-1} f_i f_{i+1}$$

is well know, and it (and its reference) can be found in the appendix.

□

The Fibonacci matrix has two equal nonzero singular values, which made the computations above possible. This is not the case for the lucas matrix. It is rank two, but its two nonzero singular values are not equal. We were able to compute them exactly. This is what we found.

Theorem 12 *[6] Assume L is the $n \times n$ Lucas matrix, $n = 2k + 1$. The two non-zero singular values of L are $5f_k f_{k+1}$ and $l_k l_{k+1}$. We have $l_k l_{k+1} \geq 5f_k f_{k+1}$ if and only if k is even, from which we conclude $||L|| = l_k l_{k+1}$ for even k, and $||L|| = 5f_k f_{k+1}$ when k is odd.*

4 Appendix

4.1 Linear algebra

We begin by recalling some undergraduate linear algebra (see [4]). We will assume that our scalars are real or complex numbers. A matrix $D = (d_{ij})$ is diagonal if all of its entries are zero, except those that occupy the northwest to southeast main diagonal: i.e. our D is diagonal when $i \neq j$ implies $d_{ij} = 0$. A diagonal matrix is called positive when the non-zero entries on the diagonal are positive numbers. A matrix U is unitary when it is a square matrix which acts *isometrically*: for each $\vec{v}$ which can be multiplied by U, we have

$$||U\vec{v}|| = ||\vec{v}||,$$

where $||\vec{v}||$ denotes the Euclidean norm of $\vec{v}$. If $\vec{v}$ has orthonormal coordinates $(v_1, \ldots, v_n)$, this Euclidean norm is the familiar

$$||\vec{v}|| = \sqrt{|v_1|^2 + \ldots + |v_n|^2}.$$

Given any matrix A, we let A^* denote the conjugate transpose of A, so if $A = (a_{ij})$, then one obtains A^* by interchanging the rows of A with its columns (transposing), then taking the complex conjugates of all the matrix entries, yielding

$$A^* = (\bar{a}_{ji}).$$

The conjugate transpose is also called the *adjoint* of A (less descriptive, but more brief). One characterization of unitary matrices is that they are invertible matrices whose adjoints are their inverses, i.e. a square matrix U is unitary if and only if $I = U^*U = UU^*$ (with I denoting the identity matrix).

Theorem 13 (Singular Value Decomposition) *Given an $n \times m$ matrix A, there exists unitary matrices U and V, and an $n \times m$ positive diagonal matrix D, such that*

$$A = UDV.$$

The non-zero entries $s_1 \geq \ldots \geq s_k > 0$ on the diagonal of D are called the *singular values* of A. The *spectral norm* of A, denoted $||A||$ is defined to be s_1. The singular values may be used to define a whole family of norms, called the Schatten p-norms, via

$$||A||_p^p = \sum s_i^p,$$

and the matrices enjoy a nice duality theory that is just like the duality theory of ℓ_p spaces. In a sense, matrices with the Schatten p-norms give a non-commutative generalization of the ℓ_p spaces. (see [8]).

4.2 Scalar identities used to obtain matrix identities

In the process of obtaining matricial Fibonacci identities, dozens of scalar identities are generated. Many of them are well known, but the literature is so vast, and the quantity of identities so large, finding and referencing a particular one is like finding a needle in a haystack. On the other hand, once an identity is written down, it is almost always quite easy to prove, using induction. We now list the identities we used, and leave all but one of the elementary induction exercises to the reader.

Our list begins with standard identities that can be found in [12], and quickly progress to the ones we need to compute matrix products. A proof of identity 4 can be found in [9]. Often the identities bounce between a form for even n and a second expression for odd n, and for this reason it is useful to incorporate the characteristic function of the even integers, which we will denote with δ, so $\delta_n = 1$ if n is even, and otherwise $\delta_n = 0$. We also include some simple identities that we found useful to prove more complicated ones to follow.

1. $\sum_{i=0}^{n} f_i^2 = f_n f_{n+1}$
2. $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$
3. $f_n f_{n-3} - f_{n-2} f_{n-1} = (-1)^n$
4. $\sum_{i=1}^{n-1} f_i f_{i+1} = f_n^2 - \delta_{n+1}$
5. $f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $l_n = \alpha^n + \beta^n$ ($\alpha > \beta$ the roots of $x^2 - x - 1$)
6. $l_i^2 + l_{i+1}^2 = 5(f_i^2 + f_{i+1}^2)$
7. $l_i^2 = 5f_i^2 + 4(-1)^i$
8. $\sum_{i=0}^{n-1} l_i^2 = 5\sum_{i=0}^{n-1} f_i^2 + 4\delta_{n-1}$
9. $\sum_{i=0}^{n-1} l_{i-1} l_i = 5 f_n f_{n-2} - 7\delta_{n-1}$
10. $\sum_{i=0}^{n-1} f_{-i} f_{n-i-1} = \sum_{i=1}^{n-2} (-1)^{i+1} f_i f_{n-i-1} = \delta_{n-1} f_{n-1}$
    (Equals f_{n-1} for odd n and it equals 0 for even n.)

11. $\sum_{i=0}^{n-2} f_{-i}f_{n-i-3} = \delta_{n-1}f_{n-1} - \delta_n f_{n-2}$
(Equals f_{n-1} for odd n and it equals $-f_{n-2}$ for even n.)

12. For $n > 2$, $\sum_{i=0}^{n-2}(-1)^i f_i f_{n-i-2} = -f_{n-2}\delta_n$
(Equals 0 for odd $n > 2$.)

13. $\sum_{i=1}^{n} f_{-i}f_{n-i} = f_n\delta_n$
(Equals 0 for odd n.)

14. $\sum_{i=1}^{n} f_{-i}f_{n-i-1} = f_{n+1}\delta_{n+1} - f_n\delta_n$
(Equals f_{n+1} for odd n and $-f_n$ for even n.)

15. $\sum_{i=-1}^{n-2} f_{-i}f_{n-i-2} = f_{n-1}\delta_{n-1} + f_n\delta_n$
(Equals f_{n-1} for odd n and f_n for even n.)

16. $\sum_{i=0}^{n-1} f_{-i}f_{n-i-2} = f_n\delta_n - f_{n-1}\delta_{n-1}$
(Equals $-f_{n-1}$ for odd n and f_n for even n.)

17. $\sum_{i=-k}^{k} f_i l_i = 0$

18. $\sum_{i=-k}^{k} f_i l_{i+1} = 5f_k f_{k+1}$

19. $\sum_{i=-k}^{k} l_i^2 = 2\sum_{i=-k}^{k} l_{i+1}l_i$

20. $\sum_{i=-k}^{k} l_{i+1}l_i = l_k l_{k+1}$

21. $l_k l_{k+1} - 5f_k f_{k+1} = 2(-1)^k$

22. $\sum_{i=0}^{n-1} l_{-i}f_{n-1-i} = 2f_n\delta_n + f_{n-1}\delta_{n-1}$

23. $\sum_{i=0}^{n-2} l_{-i}f_{n-3-i} + f_{n-2} = 2f_n\delta_n - f_{n-3}\delta_{n-1}$

24. $f_{n-1} + \sum_{i=0}^{n-2} l_{-i}f_{n-2-i} = f_n\delta_n + 3f_{n-1}\delta_{n-1}$

25. $-\sum_{i=0}^{n-1} l_{-i}f_{n-2-i} = f_n\delta_n - f_{n+2}\delta_{n-1}$

Proof of Identity 10: The first equality follows from $f_{-i} = (-1)^{i+1}f_i$. When $n = 3, 4,$ and 5 the second equality reads $f_1 = f_2$, $f_1f_2 - f_2f_1 = 0$, and

$$f_1f_3 - f_2^2 + f_3f_1 = f_4,$$

all true statements. Assume that

$$\sum_{i=1}^{n-2}(-1)^{i+1}f_i f_{n-i-1} = f_{n-1}$$

for some odd n and

$$\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i}=0.$$

Adding the two equations, we obtain

$$\begin{aligned}\textstyle\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1}+ \quad & \textstyle\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i}\\ & =\textstyle\sum_{i=1}^{n-2}(-1)^{i+1}f_i(f_{n-i-1}+f_{n-i})-f_{n-1}\\ & =\textstyle\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1}-f_{n-1}\\ & =f_{n-1}.\end{aligned}$$

The last equality implies $\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1}=2f_{n-1}$, hence

$$\sum_{i=1}^{n}(-1)^{i+1}f_if_{n-i+1}=2f_{n-1}-f_{n-1}f_2+f_nf_1=f_{n+1}.$$

Next, assume that

$$\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1}=0$$

with n even, and

$$\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i}=f_n.$$

Adding the equations gives us

$$\begin{aligned}\textstyle\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i-1}+ \quad & \textstyle\sum_{i=1}^{n-1}(-1)^{i+1}f_if_{n-i}\\ & =\textstyle\sum_{i=1}^{n-2}(-1)^{i+1}f_i(f_{n-i-1}+f_{n-i})+f_{n-1}\\ & =\textstyle\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1}+f_{n-1}\\ & =f_n.\end{aligned}$$

This time we have $\sum_{i=1}^{n-2}(-1)^{i+1}f_if_{n-i+1}=f_n-f_{n-1}$, and

$$\sum_{i=1}^{n}(-1)^{i+1}f_if_{n-i+1}=(f_n-f_{n-1})+f_{n-1}f_2-f_nf_1=0.$$

References

[1] M. Bahsi and S. Solak, On the spectral norms of Hankel matrices with Fibonacci and Lucas numbers, *Selcuk J. Appl. Math.*, **12** no. 1 (2011), 71-76.

[2] D. Bozkurt and M. Akbulak, On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers, *Hacet. J. Math. Stat.*, **37** (2)(2008), 89-95.

[3] D. Bozkurt and S. Solak On the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices, *Appl. Math. Comput.*, **140** (2003), 231-238.

[4] O. Bretscher, *Linear Algebra with Applications*, 5th edition, Pearson, 2013.

[5] D. Kalman and R. Mena, The Fibonacci numbers – exposed, *Mathematics Magazine*, vol. **76**, no. 3 (2003)

[6] J. Dixon, B. Mathes, and D. Wheeler, An application of matricial Fibonacci identities to the computation of spectral norms, *Rocky Mt. J. Math.*, to appear.

[7] E. Dupree and B. Mathes, Singular values of k-Fibonacci and k-Lucas Hankel Matrices, *Int. J. Contemp. Math. Sciences*, Vol. 7, 2012, no. 47, 2327 - 2339

[8] Pedersen, Gert K., *Analysis Now*, Graduate Texts in Mathematics **118**, Spriger-Verlag, (1989).

[9] S. Shen, On the norms of Toeplitz matrices involving k-Fibonacci and k-Lucas numbers, *Int. J. Contemp. Math. Sciences*, **7** (8) (2012), 363-368.

[10] S. Shen, J. Cen, On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers, *Int. J. Contemp. Math. Sciences*, **5** (12) (2010), 569-578.

[11] S. Solak, On the norms of circulant matrices with the Fibonacci and Lucas numbers, *Appl. Math. Comput.*, **160** (2005), 125-132.

[12] E. W. Weisstein, *CRC Encyclopedia of Mathematics*, CRC Press, Chapman and Hall, vol. 1 (2009).