2015 HAWAII UNIVERSITY INTERNATIONAL CONFERENCES
S.T.E.A.M. & EDUCATION JUNE 13 - 15, 2015
ALA MOANA HOTEL, HONOLULU, HAWAII
S.T.E.A.M & EDUCATION PUBLICATION:
ISSN 2333-4916 (CD-ROM)
ISSN 2333-4908 (ONLINE)

CONQUERING PERFECT NUMBERS USING THE TOOL OF FINITE SERIES

MULATU, LEMMA & ET AL
SAVANNAH STATE UNIVERSITY, GEORGIA
DEPARTMENT OF MATHEMATICS

Prof. Lemma Mulatu
Dr. Jonathan Lambright
Dr. Ayana Tiller
Department of Mathematics
Savannah State University
Georgia.

Conquering Perfect Numbers Using the Tool of Finite Series

Synopsis:

Mathematicians and non-mathematicians have been fascinated for centuries by the properties and patterns of numbers. They have noticed that some numbers are equal to the sum of all of their factors (not including the number itself). Such numbers are called perfect numbers. Thus a positive integer is called a perfect number if it is equal to the sum of its proper positive divisors. The search for perfect numbers began in ancient times.

Conquering Perfect Numbers Using the tool of finite series

Mulatu Lemma, Jonathan Lambright and Ayana Tiller
Department of Mathematics
Savannah State University
Savannah, Georgia 31404
Hawaii University International Conference

Abstract. Mathematicians have been fascinated for centuries by the properties and patterns of numbers. They have noticed that some numbers are equal to the sum of all of their factors (not including the number itself). Such numbers are called perfect numbers. Thus a positive integer is called a perfect number if it is equal to the sum of its proper positive divisors. The search for perfect numbers began in ancient times. The four perfect numbers 6, 28, 496, and 8128 seem to have been known from ancient times. In this paper, we will investigate some important properties of perfect numbers. We give easy and simple proofs of theorems using finite series. We give our own alternative proof of the well-known Euclid's Theorem (Theorem I). We will also prove some important theorems which play key roles in the mathematical theory of perfect numbers..

Key Words: Prime Numbers, Perfect numbers, and Triangular numbers.

1. Introduction and Background

Throughout history, there have been studies on perfect numbers. It is not known when perfect numbers were first studied and indeed the first studies may go back to the earliest times when numbers first aroused curiosity [6]. It is rather likely, although not completely certain, that the Egyptians would have come across such numbers naturally given the way their methods of calculation worked, where detailed justification for this idea is given [6]. Perfect numbers were studied by Pythagoras and his followers, more for their mystical properties than for their number theoretic properties [6]. Although, the four perfect numbers 6, 28, 496 and 8128 seem to have been known from ancient times and there is no record of these discoveries [6]. The First recorded mathematical result concerning perfect numbers which is known occurs in Eculid's Elements written around 300BC [6]

2. The Main Results

Proposition 1: If 2^n-1 is prime, then n is prime for n>1.

Proof: Suppose *n* is not prime, then there exist positive integers a and b $n = a \times b, a > 1, b > 1$.

Then, $2^n - 1 = 2^{ab} - 1 = \left(2^a - 1\right)\left(2^{a(b-1)} + 2^{a(b-2)} + 2^{a(b-3)} ... + 2^a + 1\right)$

Since 2^n-1 is prime and $\left(2^{a(b-1)} + 2^{a(b-2)} + 2^{a(b-3)} ... + 2^a + 1\right) > 1$, it follows that

$$2^a - 1 = 1$$
$$\Rightarrow \quad 2^a = 2^1$$
$$\Rightarrow \quad a = 1$$

This is a contradiction, hence *n* is prime.

Remark I: The converse to proposition 1 is false as *n* = 11 is prime but 2^{11}-1 is not prime as 2^{11}-1 = 23× 89

Theorem 1 . If 2^k -1 (k>1) is prime, then $n = 2^{k-1}(2^k-1)$ is a perfect number.

Proof: We will show that n = sum of its proper factors.
We will find all the proper factors of $2^{k-1}\left(2^k - 1\right)$, and add them.
Since 2^k-1 is prime, let $p = 2^k-1$. Then n = p(2^k-1)

Let us list all factors of 2^{k-1} and other proper factors of n as follows .

Factors of 2*k*-1	Other Proper Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
.	.
2^{k-1}	$2^{k-2}p$

Adding the first column, we get:

$$1 + 2 + 2^2 + 2^3 ... + 2^{k-3} + 2^{k-2} + 2^{k-1}$$
$$= 2^k - 1$$
$$= p$$

Adding the second column, we get:

$$p + 2p + 2^2 p + 2^3 p \ldots + 2^{k-4} p + 2^{k-3} p + 2^{k-2} p$$
$$= p\left(1 + 2 + 2^2 + \ldots + 2^{k-2}\right)$$
$$= \left(2^{k-1} - 1\right) p$$

Now adding the two columns together, we get:

$$\begin{aligned} & p + p\left(2^{k-1} - 1\right) \\ &= p\left(1 + 2^{k-1} - 1\right) \\ &= p\left(2^{k-1}\right) \\ &= n \end{aligned}$$

Hence. n is a perfect number.

Remark II: A question can be raised if k is prime by itself

$\Rightarrow$ $2^{k-1}(2^k-1)$ is a perfect number. The answer is negative as it will be easily shown that it does not work for k=11.

Corollary 1: If 2^k-1 is prime, $1+2+3+4...+2^k$ -1 is a perfect number.

Proof: Note that:

$$\begin{aligned} n &= 1+2+3+4 \;...+2^k - 1 \\ &= \frac{\left(2^k - 1 + 1\right)\left(2^k - 1\right)}{2} \\ &= \frac{2^k\left(2^k - 1\right)}{2} \\ &= 2^{k-1}\left(2^k - 1\right) \end{aligned}$$

$\Rightarrow$ *n* is a perfect number by Theorem 1.

Corollary II: If 2^k-1 is prime, then $n = 2^{k-1} + 2^k + 2^{k+1} ... + 2^{2k-2}$ is a perfect number.

Proof: We have:

$$n = 2^{k-1} + 2^k + 2^{k+1} ... + 2^{2k-2} = 2^{k-1}\left(1 + 2 + 2^2 + 2^3 ... + 2^{k-1}\right)$$

$$n = 2^{k-1}\left(2^k - 1\right)$$

$\Rightarrow$ *n* is a perfect number by Theorem 1.

Remark III: Every even perfect number *n* is of the form $n = 2^{k-1}(2^k-1)$. We will not prove this, but we will accept and use it.
So, the converse to Theorem 1 is also true. This is called Euler's Theorem.

Next we will show how **Remark III** applies to the first four perfect numbers. Note that:

$$6 = 2 \cdot 3 = 2^1\left(2^2 - 1\right) = 2^{2-1}\left(2^2 - 1\right)$$
$$28 = 4 \cdot 7 = 2^2\left(2^3 - 1\right) = 2^{3-1}\left(2^3 - 1\right)$$
$$496 = 16 \cdot 31 = 2^4\left(2^5 - 1\right) = 2^{5-1}\left(2^5 - 1\right)$$
$$8128 = 64 \cdot 127 = 2^6\left(2^7 - 1\right) = 2^{7-1}\left(2^7 - 1\right)$$

Theorem II. Every even perfect number *n is* a triangular number.

Proof: n is a perfect number $\Rightarrow$ n= $2^{k-1}(2^k-1)$ by Remark III. Hence,
n= $\frac{2^k(2^k-1)}{2} = \frac{(m+1)m}{2}$, where m=$2^k$-1 . Thus n is a triangular number.

Corollary III If T is a perfect number, then 8T +1 is a perfect square**.**

Proof: T is a perfect number$\Rightarrow$ T is a triangular number.
$\Rightarrow$ T=$\frac{(m+1)m}{2}$ for some positive integer m.
$\Rightarrow$ 8T+1 = $4m(m+1)$ +1
= $4m^2$ +4m+1
=$(2m+1)^2$

Next we will prove two important theorems which play key roles in our study of perfect numbers .

Theorem III: The sum of the reciprocals of the factors of a perfect number is *n* is equal to 2.

Proof: Let $n = 2^{k-1}(2^k-1)$ where $p = 2^k-1$ and is prime. Let us list all the possible factors of *n*.

Factors of 2^{k-1}	Other Factors
1	p
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-1}p$

Sum of reciprocals of factors of 2^{k-1}

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}...+\frac{1}{2^{k-1}}$$

$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}}{2(2^{k-1})}+\frac{2^{k-1}}{2^2(2^{k-1})}...+\frac{1}{(2^{k-1})}$$

$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-1}\cdot 2^{-1}}{2^{k-1}}+\frac{2^{k-1}\cdot 2^{-2}}{2^{k-1}}...+\frac{1}{2^{k-1}}$$

$$=\frac{2^{k-1}}{2^{k-1}}+\frac{2^{k-2}}{2^{k-1}}+\frac{2^{k-3}}{2^{k-1}}...+\frac{1}{2^{k-1}}$$

$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}...+1}{2^{k-1}}$$

$$=\frac{2^k-1}{2^{k-1}}=\frac{p}{2^{k-1}}$$

Sum of reciprocals of other factors

$$\frac{1}{p}+\frac{1}{2p}+\frac{1}{2^2 p}+\frac{1}{2^3 p}...+\frac{1}{2^{k-1}p}$$

$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}}{2(2^{k-1}p)}+\frac{2^{k-1}}{2^2(2^{k-1}p)}...+\frac{1}{(2^{k-1}p)}$$

$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-1}\cdot 2^{-1}}{2^{k-1}p}+\frac{2^{k-1}\cdot 2^{-2}}{2^{k-1}p}...+\frac{1}{2^{k-1}p}$$

$$=\frac{2^{k-1}}{2^{k-1}p}+\frac{2^{k-2}}{2^{k-1}p}+\frac{2^{k-3}}{2^{k-1}p}...+\frac{1}{2^{k-1}p}$$

$$=\frac{2^{k-1}+2^{k-2}+2^{k-3}...+1}{2^{k-1}p}$$

$$=\frac{2^k-1}{2^{k-1}p}=\frac{p}{2^{k-1}p}=\frac{1}{2^{k-1}}$$

Now the sums of reciprocals of all factors are equal to:

$$= \frac{p}{2^{k-1}} + \frac{1}{2^{k-1}}$$

$$= \frac{p+1}{2^{k-1}}$$

$$= \frac{2^k - 1 + 1}{2^{k-1}}$$

$$= \frac{2^k}{2^{k-1}} = 2$$

<u>Corollary IV</u>: No proper divisor of a perfect number can be perfect.

Proof: Suppose n is a perfect number and d its proper divisor. Let 1, x, x_2, $x_3 \ldots x_m$, d be divisors of d.

If d is a perfect number, then $1 + \frac{1}{x} + \frac{1}{x_2} + \frac{1}{x_3} ... + \frac{1}{x_m} + \frac{1}{d} = 2$, but this is not possible as 1, x, x_2, $x_3 \ldots x_m$, d are also factors of n and $1 + \frac{1}{x} + \frac{1}{x_2} + \frac{1}{x_3} ... + \frac{1}{x_m} + \frac{1}{d}$+(*sum of the reciprocals of other remaining factors*)=2. Hence, $1 + \frac{1}{x} + \frac{1}{x_2} + \frac{1}{x_3} ... + \frac{1}{x_m} + \frac{1}{d} \neq 2$ and is not perfect.

To clearly understand Example 3, let us take n = 8128 and d = 508.

Factors of 508 are 1, 2, 4, 127, 254, and 508. Now observe that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{127} + \frac{1}{254} + \frac{1}{508}$$

$$= \frac{508 + 254 + 127 + 4 + 2 + 1}{508}$$

$$= \frac{896}{508} \neq 2$$

Factors of 8128 are: 1, 2, 4, 8, 16, 32, 64,127, 254, 508, 1016, 2032, 4064, and 8128. Note that

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{127}+\frac{1}{254}+\frac{1}{508}+\frac{1}{1016}+\frac{1}{2032}+\frac{1}{4064}+\frac{1}{8128}$$
$$=\frac{8128+4064+2032+1026+508+254+127+64+32+16+8+4+2+1}{8128}$$
$$=\frac{16256}{8128}=2$$

Corollary V . No power of a prime can be a perfect number.

Proof: . Let p be prime and let $n = p^k$. The factors of n are 1, p, p^2, $p^3 \ldots p^k$.
Now, we have:

$$1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}\ldots+\frac{1}{p^k}$$
$$=1+\frac{p^{k-1}+p^{k-2}+p^{k-3}\ldots+p+1}{p^k}$$
$$=1+\frac{p^k-1}{p^k(p-1)}$$
$$\leq 1+\frac{p^k-1}{p^k}$$
$$=1+\frac{p^k}{p^k}-\frac{1}{p^k}$$
$$=1+1-\frac{1}{p^k}$$
$$=2-\frac{1}{p^k}<2.$$

Therefore, n is not a perfect number.

Theorem IV: If n is a perfect number such that $n = 2^{k-1}(2^k-1)$, then the product of the positive divisor's of n is equal to n^k.

Proof: We list factors of n as in Theorem 2

Factors of 2^{k-1}	Other Factors
1	*P*
2	$2p$
2^2	2^2p
2^3	2^3p
:	:
:	:
2^{k-1}	$2^{k-1}p$

Product of column 1 =

$$1*2*2^2*2^3...*2^{k-1} = 2^{1+2+3...+(k-1)} = 2^{\frac{k(k-1)}{2}}$$

Product of column 2 =

$$p \cdot 2p \cdot 2^2 p ... \cdot 2^{k-1} p$$
$$= p^k (1 \cdot 2 \cdot 2^2 \ldots 2^{k-1})$$
$$= p^k (2^{\frac{k(k-1)}{2}}),$$

Therefore the products of both columns are

$$= 2^{\frac{k(k-1)}{2}} \cdot p^k \cdot 2^{\frac{k(k-1)}{2}}$$
$$= 2^{k(k-1)} \cdot p^k$$
$$= (2^{k-1} \cdot p)^k$$
$$= n^k.$$

<u>Example 1</u>: Apply Theorem IV to $n = 28$

$n = 28 = 2^2(2^3-1)$ (Here $k = 3$)

Factors of 28 are 1, 2, 4, 7, 14, and 28

The product of the factors of 28 =

$$1 \cdot 2 \cdot 4 \cdot 7 \cdot 14 \cdot 28$$
$$= 28 \cdot 28 \cdot 28$$
$$= 28^3$$

<u>Example 2</u>: Apply Theorem IV to $n = 496$

$n = 496 = 2^4(2^5-1)$ (Here $k = 5$)

Factors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 496

The products of the factors of 496 =

$$1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 31 \cdot 62 \cdot 124 \cdot 248 \cdot 496$$
$$= 496 \cdot 496 \cdot 496 \cdot 496 \cdot 496$$
$$= 496^5$$

Hunt for Some Perfect Numbers Using Calculator

$n = 2^{k-1}(2^k - 1)$ (2^k-1 is prime)

$p_1 = 2^1(2^2-1) = 6$ (k = 2)

$p_2 = 2^2(2^3-1) = 28$ (k = 3)

$p_3 = 2^4(2^5-1) = 496$ (k = 5)

$p_4 = 2^6(2^7-1) = 8128$ (k = 7)

$p_5 = 2^{12}(2^{13}-1) = 33550336$ (k = 13)

$p_6 = 2^{16}(2^{17}-1) = 8589869056$ (k = 17)

$p_7 = 2^{18}(2^{19}-1) = 137438691328$ (k = 19)

$p_8 = 2^{30}(2^{31}-1) = 2305843008139952128$ (k = 31)

$p_9 = 2^{60}(2^{61}-1) = 2658455991569831744654692615953842176$ (k = 61)

$p_{10} = 2^{88}(2^{89}-1) =$
191561942608236107294793378084303638130997321548169216
(k = 89)

$p_{11} = 2^{106}(2^{107}-1) =$
1316403645856964833723975346045872291022347231838694311778 3
728128 (k = 107)

$p_{12} = 2^{126}(2^{127}-1) =$
14474011154664524427946373126085988481573677491478358890663
54349131199152128 (k = 127)

$p_{13} = 2^{520}(2^{521}-1) =$
23562723457267347065789548996709904988477547858392600710143
02759750633728317862223973036553960260056136025556646250327
01750528925780432155433824984287771524270103944969086640286
44534128033831439790236838624033171435922356643219703101720
71316352748729874740064780193958716593640108741937564905791
8549492160555646976 (k = 521)

$p_{14} = 2^{606}(2^{607}-1) =$
14105378370671206906320795808606318988148674351471566783883
86759999548677426523801141041933290376902515619505687098293
27164087724366370087116731268159313652487450652439805877296
20729744672329516665822884692680778665287018892086787945147
83645693139220603706950647360735723786951764730552668262532
84886383715072974324463835300053138429460296575143368065570
759537328128 (k = 607)

Open Questions

We were able to observe that there are open questions concerning perfect numbers which can be excellent potential research problems for future work for all interested mathematicians. The following are the open questions which are potential research problems to work on.

1. Are there any odd perfect numbers or are all perfect numbers even?
2. Is there a finite amount of perfect numbers or are there infinitely many?

We hope that sometime in the near future these questions will be answered as the famous Fermat's Last Theorem did.

Acknowledgements

We would like to thank Drs. Lemma and Chetty for their excellent encouragement and support. We are indebted to the editor, referees for their great assistant. ,Also we would like to thank the MAGEC-STEM program, funded by the National Science Foundation, at Savannah State University for providing us with this golden opportunity which could play a big role in our future study of mathematics.

References

1. Burton, D. M. (1998). *Elementary number theory*. New York City, New York: McGraw-Hill.

2. Dodge, C. W. (1975). *Numbers and mathematics*. Boston, Massachusetts: Prindle, Weber & Schmidt Inc..

3. Dudley, Underwood (1969). *Elementary number theory*. San Francisco: W. H. Freeman and Company.

4. Jackson, T. H. (1975). *Number theory*. Boston, Massachusetts: Rouledge & Kegan Paul Ltd..

5. Mersenne Prime. (2006). *Wikipedia*. Wikimed Foundation Icn. Retrieved July 12, 2006. http://en.wikipedia.org/wiki/Mersenne_prime

6. O'Conner, J.J. E.F. Robertson. (2001, December). Perfect number's. Retrieved 7/6/2006. http://www.history.mcs.st-andrews.ac.uk/HistTopics/Perfect_numbers.htm

7. Perfect Number. (2006). *Wikipedia*. Wikimed Foundation Icn. Retrieved July 12, 2006. http://en.wikipedia.org/wiki/Perfect_number

8. Shanks, D. (1978). *Solved and unsolved problems in number theory*. New York City, New York: Chelsea Publishing Company.

9. Spencer, D. D. (1995). *Key dates in number theory history: from 10,529 B.C. to the present*. Ormond Beach, Florida: Camelot Publishing Company.

10. Weisstein, Eric W. "Perfect Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PerfectNumber.html