# Excursions in the Predecessor Trees of the Accelerated Collatz Map

Andrey Rukhin[1,*]

*Naval Surface Warfare Center — Dahlgren Division*
*Dahlgren, Virginia, 22448 USA*

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**Abstract**

Building on the result in [6], we examine the structure that emerges within the predecessor structure of the *accelerated Collatz map* $C \colon \mathbb{N} \to \mathbb{N}$

$$C(n) = \frac{3n+1}{2^M}$$

where $n \geq 1$ and the integer $M$ is such that $2^{M+1} \nmid (3n+1)$.

The inverse maps of both the "classic" and "accelerated" Collatz function have been investigated by numerous authors. The "chalice" tree structure has been studied in [1] and [5] and certain numeric properties are identified for a few preimage iterates. In [2], the authors provide algorithms for computing $3x+1$ inverse trees. The authors in [3] derive formulas for the expected number of leaves in a $3x+1$ tree of depth $k$ whose root satisfies certain criteria. The author in [4] poses a conjecture regarding the nature of certain preimage sequences within the accelerated map. The author in [7] considers certain properties of preimage sequences (modulo 2) in the classic problem.

Assume $g$ is an odd positive integer. We define the *j-predecessor set* of $g$ to be

$$\pi^{(j)}(g) = \{l \text{ odd} \mid C^{(j)}(l) = g\}.$$

We'll inductively define a rooted tree structure $T(g)$ with labelled vertices as follows. We'll say that the root of the tree has label $g$ and we'll define the *height* of the root to be 0. For any vertex $v$ of height $i$ ($i \geq 0$) and label $m$, we'll say that the vertex $u$ (with label $n$) is a predecessor of $v$ if $n \in \pi^{(1)}(m)$. We will also define the height of $u$ to be $i+1$.

We define a *vein at m* to be the vertices of the form

$$V_m = \{v_{b(j,m)} \mid j \geq 0\}$$

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*Phone: 011-540-6539204, Fax: 011-540-6532641 , Postal Address: 18444 Frontage Road, Bldg. 1470, Suite 327, Dahlgren, Virginia 22448 USA

*Email address:* `andrey.rukhin@navy.mil` (Andrey Rukhin)

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where $b(j,m) = 4^j m + \sum_{i=0}^{j-1} 4^i$ and we define $v_m$ to be the *head* of the vein.

We will define the *j-predecessor* tree of $g$, denoted by $T^{(j)}(g)$, to be the subtree of $T(g)$ over the vertices of distance $i$ ($i \leq j$) from the root $g$. We define a $j$-predecessor branch in $T^{(j)}(g)$ to be a sequence $(g_i)_{i=0}^{j}$ where $g_0 = g$ and $C(g_i) = g_{i-1}$ for $i \in [j]$.

We will also introduce a coloring $c \colon \mathbb{N} \to \{0,1,2\}$ defined as

$$c(v_g) = g \bmod 3$$

for all odd integers $g$. With $g$ and $j$ as above, we'll denote the colored predecessor trees by $\tau(g)$ and $\tau^{(j)}(g)$, respectively.

Consider the illustrated examples in Figure 1. We view a path of vertices connected by solid edges to be veins, and we represent their common successor to be connected to the head of the vein. In this example, one observes that the three predecessor trees are isomorphic (modulo 3) up to height 2 (and each of the corresponding branch colors differ at height 3).

The goal of this talk is to identify the permutations of strings over the set $\{0,1,2\}^j$ that naturally arise within the colored predecessor trees of height $j$. We detail some interesting characteristics of these ternary sequences, and we outline an inductive construction of these permutations of the set $\{0,1,2\}^j$. We also connect these permutations to the conjecture of Banerji [4].

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[1] Sergio Albeverio, Danilo Merlini, and Remiglio Tartini. Una breve introduzione a diffusioni su insiemi frattali e ad alcuni essempi di sistemi dinamici semplici. *Note di matematica e fisica, Edizioni Cerfim Locarno*, 3:1–39, 1989.

[2] Stefan Andrei and Cristian Masalagiu. About the collatz conjecture. *Acta Informatica*, 35:167–179, 1998.

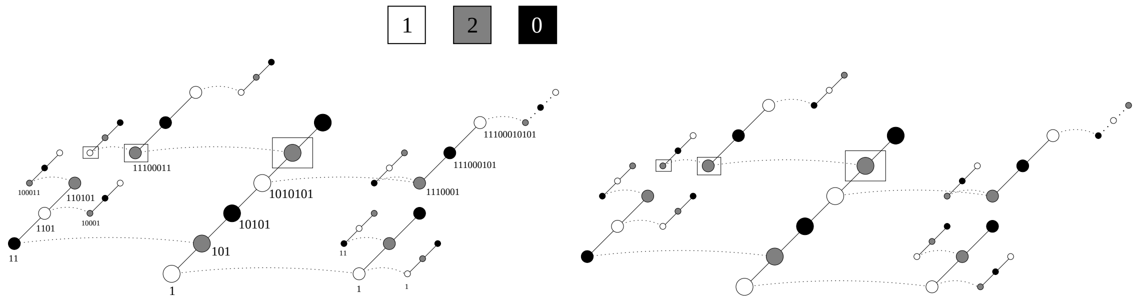
[3] David Applegate and Jeffrey C. Lagarias. On the distribution of 3x + 1 trees. *Experimental Mathematics*, 4:101–117, 1995.

[4] Ranan B. Banerji. Some properties of the 3n + 1 function. *Cybernetics and Systems*, 27:473–486, 1996.

[5] Stefano Beltraminelli, Danilo Merlini, and Luca Rusconi. Orbite inverse nel problema del 3n + 1. *Note di matematica e fisica, Edizioni Cerfim Locarno*, 7:325–357, 1994.

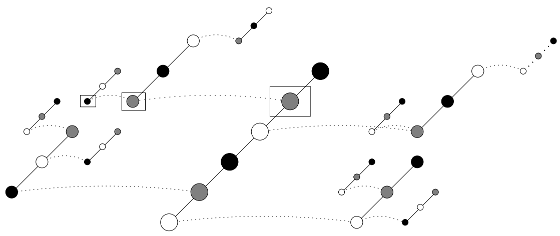
[6] Andrey Rukhin. On a cyclic property in the predecessor trees of the accelerated collatz map. *Submitted*, 2011.

[7] Gunther J. Wirsching. On the combinatorial structure of 3n + 1 predecessor sets. *Discrete Math.*, 148:265—286, 1996.




(a) Illustration of the first few vertices in $V_1$.

(b) Illustration of the first few vertices in $V_{1(01)^9}$.



(c) Illustration of the first few vertices in $V_{1(01)^{18}}$.

Figure 1: Few of the vertices in the first tree are labelled with odd (binary) integers. The boxed vertices in each tree highlight three branches (rooted at their central paths) whose colors agree up to and excluding height 3 (the reader can convince himself that this property holds for any triple of corresponding branches). These branches give rise to ternary sequences (e.g. the boxed branches yield the sequences 221, 222 and 220) and the goal of this talk is to identify the structure of these the ternary sequences.