DYNAMIC PLANES OF TANGENT-LIKE MEROMORPHIC FUNCTIONS

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ABSTRACT. The theory of iterated transcendental functions and entire functions has been extensively studied in the past two decades. In particular, the dynamical planes of exponential function and tangent function have garnered much attention. In this paper, we study the relationship between the dynamical planes of these two families. We investigate the family of meromorphic functions with two asymptotic values, which we call tangent-like functions. We show that there is dynamical convergence of tangent-like functions to the exponential function.

1. Introduction

In recent years, there has been increasing interest in the study of the transcendental entire and meromorphic functions (see, e.g., [1,2, 20, 21,31,32]). In particular, the exponential functions $\lambda\exp(z)$ with $\lambda \in C \setminus \{0\}$ have been most extensively studied among the entire functions. One reason is that $\lambda\exp(z)$ is the simplest family because $\lambda\exp(z)$ has only one singular value 0 which corresponds to the quadratic family $z^2 + C$ which has one critical value. In this paper, we are interested in the meromorphic functions with two asymptotic values. One simple example is the family $\lambda\tan(z)$ which has two symmetric asymptotic values λ and $-\lambda$. It has been studied by several authors. Here we study the family with two asymptotic values where one is fixed at $-\lambda$ and the other one at $a\lambda$. Under the conjugation the family can be written as $T_{a,\lambda}(z) = a\lambda\frac{\exp(z)-\exp(-z)}{\exp(z)+a\exp(-z)}$, and we can see as a approaches ∞, the asymptotic value a escapes to ∞, which on any compact subset, uniformly converges to the exponential family $\lambda\exp(2z) - \lambda$.

It is natural to ask about the relation between the Julia sets of two families. We show that there is dynamic convergence as $a \to \infty$, and also we study the relationship of the hyperbolic components of two families. We are interested in the dynamic planes of $T_{a,1}$ and $E(z)$. These are the real slices of the families $T_{a,\lambda}$ and E_λ when $\lambda = 1$. We are interested in the relationship between the functions in the two families. We have the following:

- When $a \to \infty$, the functions $T_a(z)$ converge uniformly on any compact subset to the exponential function $E(z)$.
- For the function $T_a(z)$, when $a > 1$, the Julia set is a curve bounded by the vertical lines l_1 and l_2, where l_1 is the line $x = 0$ and l_2 is the line $x = \frac{1}{2}\log(a)$. The Julia set passes through the points $z = n\pi i$ on l_1 and the points $z = \frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ on l_2, for all n. The Fatou set consists of two completely invariant components, each of which contains one of the asymptotic values.

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When $a = 1$, the function $T_a(z) = \tanh(z)$ is conjugate to the tangent function $\tan(z)$ and the Julia set is the imaginary axis $x = 0$.

When $0 < a < 1$, the Fatou set consists of one completely invariant component containing both asymptotic values, and the Julia set of $T_a(z)$ is a Cantor set.

- We prove that the Julia set of $T_{a,1}$ converges to the Julia set of the exponential function $E(z)$, which is a Cantor bonquet.

For any fixed real λ, we study the relationship of the dynamic planes of $T_{a,\lambda}$ and $E_\lambda(z)$, and we also describe the Symbolic Dynamics for $T_{a,\lambda}$ and E_λ, and prove that $T_{a,\lambda}$ converges to E_λ in the combinatorial sense.

We study the parameter plane of $\mathcal{T}_{a,\lambda}$, when a is real, and discuss the relationship of $\mathcal{T}_{a,\lambda}$ to the tangent family $\mathcal{T}_\lambda$ when a is near 1. We also study the relationship of $\mathcal{T}_{a,\lambda}$ to the exponential family $\mathcal{E}_\lambda$ when a is near ∞. When $a \to \infty$, the hyperbolic components of $\mathcal{T}_{a,\lambda}$ approach the hyperbolic components of $\lambda \exp(z)$. That is, for a λ, such that $\exp_\lambda(z)$ has a periodic cycle of period p, there exists a sequence λ_i, $i = 1, 2, \cdots, n, \cdots$, such that $T_{a_{\lambda_i},\lambda_i}(z)$ has a periodic cycle of period p, satisfying $\lambda_i \to \lambda$ and $a_{\lambda_i} \to \infty$ as $i \to \infty$.

This paper is organized as follows: In section 2 we introduce the notations used throughout the paper and list some definitions and basic facts about the dynamics of meromorphic functions in general. In section 3 we discuss the Dynamical planes of $T_a(z)$ and $E(z)$. In section 4 we discuss Symbolic Dynamics for $T_{a,\lambda}$ and E_λ and prove the convergence in the combinatorial sense. In section 5, we study the dynamic plane of functions in the family $\mathcal{T}_{a,\lambda}$ when both a and λ are real and prove the dynamic convergence to the exponential function as a approaches ∞. In section 6 we discuss the parameter plane of the family $\mathcal{T}_{a,\lambda}$ for a fixed a which is real. We also discuss the relationship between the parameter planes of the family $\mathcal{T}_{a,\lambda}$ and the exponential family $\mathcal{E}_\lambda$.

2. Notations and Preliminaries

Throughout this paper, we use the following notations.

- $\mathcal{E}_\lambda$ stands for the exponential family $\{E_\lambda(z) = \lambda \exp(2z) - \lambda, \lambda \in \mathbb{C} \setminus \{0\}\}$.
- $\mathcal{T}_\lambda$ stands for the tangent family $\{T_\lambda = \lambda \tan(z), \lambda \in \mathbb{C} \setminus \{0\}\}$.
- $\mathcal{T}_a$ stands for the family with two asymptotic values with the normal form $\{T_a = a\frac{\exp(z)-\exp(-z)}{\exp(z)+a\exp(-z)}, \lambda \in \mathbb{C} \setminus \{0\}\}$.
- $\mathcal{T}_{a,\lambda}$ stands for the family with two asymptotic values with the normal form $\{T_{a,\lambda} = a\lambda\frac{\exp(z)-\exp(-z)}{\exp(z)+a\exp(-z)}, \lambda \in \mathbb{C} \setminus \{0\}\}$.

Julia sets and Fatou sets Let us recall some definitions and basic facts about the dynamics of meromorphic functions in general. For a complex analytic function f, the Julia set carries the interesting dynamical information (See, [5,7,8,26]).

Definition 1. The *Fatou set* of f is defined to be: $\{z : z \in \hat{\mathbb{C}}$ and there exists a neighborhood U of z such that all f^n, $n \in \mathbb{N}$, is defined in U and form a normal family in $U\}$. The *Julia set* $J(f)$ is the complement of the Fatou set.

For a function a meromorphic function f the Julia set $J(f)$ can also be described by the following equivalent conditions:

- $J(f)$ is the closure of the set of repelling periodic points of f.

- There are points whose forward orbit lands on a pole and are thus preimages of ∞. Such points are called prepoles and $J(f)$ is the closure of the set of prepoles of f.

Cantor Sets and Cantor Bouquets The Julia set of some rational functions and transcendental meromorphic functions is homeomorphic to the standard Cantor set. Topologically, a Cantor set is non-empty, perfect, compact, totally disconnected, and metrizable. A perfect set is a closed set in which every point is an accumulation point.

A Cantor bouquet appears in the Julia set of certain entire functions. Roughly speaking, a Cantor bouquet can be thought of as a collection of cantor sets of curves. Topologically, it can be defined as follows (See [8,14]) :

Definition 2. A *Cantor bouquet* is a subset of C which is homeomorphic to a straight brush. By a straight brush, we mean that a subset $\mathbf{S}$ of $[0,\infty)\times(\mathbb{R}-\mathbb{Q})$ has the following properties:

- For every $s\in\mathbb{R}-\mathbb{Q}$, there is a $t_s\in[0,\infty)$ such that all points $(t,s)\in\mathbf{S}$ where $t\geqslant t_s$. The hair h_s is defined by $h_s=[t_s,\infty)\times\{s\}$. The point $e_s=(t_s,s)$ is called the endpoint of the hair h_s.
- The set $\{\alpha|(y,\alpha)\in\mathbf{S}$ for some $y\in[0,\infty)\}$ is dense in $\mathbb{R}-\mathbb{Q}$. Moreover, each endpoint of a hair is the limit from above and from below of other endpoints of hairs.
- $\mathbf{S}$ is closed in $\mathbb{R}^2$.

Singular Values, Critical Values, and Asymptotic Values As we know, studying the orbit of the critical points or values gives us a lot of information about the dynamics of the rational functions. Similarly, by studying what happens to the singular values of transcendental entire or meromorphic functions, we garner much information about the dynamics of transcendental functions.

Definition 3. Suppose that $f:D\to\hat{\mathbb{C}}$ is a nowhere locally constant holomorphic function where D is open and $D\subset\hat{\mathbb{C}}$. A point $u\in\hat{\mathbb{C}}$ is said to have a regular covering if it has a neighborhood U such that each component $f^{-1}(U)$ is mapped holomorphically onto U by f. A *singular value* is a value at which f is not a regular covering.

Definition 4. A point z is called a *critical value* if there exists a point $w\in f^{-1}(z)$, a neighborhood U of z, and a neighborhood W of w such that $f:U\setminus\{z\}\to W\setminus\{w\}$ is a degree-d covering for some $d>1$. w is called a critical point.

Definition 5. A point v is called an *asymptotic value* if there exists a path $\alpha:[0,\infty)\to D$ such that $\alpha(t)\to\partial D$ and $f\circ\alpha(t)\to v$ as $t\to\infty$.

Notice that if v is isolated in the singular set then we can choose a small neighborhood V of v whose closure contains no other singular values. Let U be the component of $f^{-1}(V)$ containing the tail of α. Then $f:U\to V\setminus\{v\}$ is a universal covering.

Definition 6. If we can associate a given asymptotic value v with an asymptotic tract, that is, a simply connected unbounded domain A such that $f(A)$ is a punctured neighborhood of v, then v is called a *logarithmic singularity.*

Definition 7. A set U is called *completely invariant* under f if $z \in U$ implies $f(z) \in U$ when $f(z)$ is defined. Also, for all w such that $f(w) = z$, $w \in U$. That is, $U \subset f(U)$ and $f^{-1}(U) \subset U$.

The following background material about Fatou and Julia sets can be found in the books (see, [4], [7], [26]) or in the paper (see, [5]).

Basic Properties of Fatou and Julia sets

Lemma 1. *The Fatou set and Julia set are completely invariant.*

Lemma 2. *Either the Julia set is $\hat{\mathbb{C}}$ or the Julia set has an empty interior.*

Definition 8. If the backward orbit of z_0 is finite, then we say that z_0 is an *exceptional point.*

Lemma 3. *If z_0 is not an exceptional point in the Julia set, then the Julia set is the closure of the backward orbit of z_0.*

Proposition 1. *If f is meromorphic then the Julia set $J(f)$ is a perfect set.*

Periodic Points and Periodic Components Periodic points play an important role in iteration theory. By definition, periodic points consist of points such that $f^n(z) = z$ for some $n \geqslant 1$ and $f^k(z) \neq z$ for any $k < n$. For a periodic point z_0 of period n, the multiplier of z_0 is defined by $(f^n)'(z_0)$. Notice that the multiplier is independent of the choice of conjugation of the function. (When $z_0 \to \infty$, the multiplier is defined as $(g^n)'(0)$; where $g(z) = 1/f(1/z)$.) A periodic point is called attracting, indifferent, or repelling when the modulus of the multiplier is less than 1, equal to 1, or greater than 1. A periodic point of period 1 is called a fixed point.

By definition, the Fatou set F is open. By convention, a component U of the Fatou set F always means a connected component of F. For a component U, $f^n(U)$ is contained in some component, which we label U_n, of F. A component U is called preperiodic if there exist $m > n \geqslant 0$ such that $U_m = U_n$. A component U is called periodic component of period n if we have $U_n = U$ for some $n > 0$. U is called invariant if $f(U) \subset U$. A component U is called a wandering domain if U is not preperiodic.

The Classification of Periodic Components Let U be a periodic component of period p, then we have one of the following:

- U is attractive: each U contains an attracting periodic point z_0 of period p. Then $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$.
- U is parabolic: the boundary of U contains a periodic point z_0 of period p. The multiplier of z equals $\exp(2\pi ip/q), (p, q) = 1$, and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$.
- U is a Siegel disk: there exists an holomorphic homeomorphism $\phi : U \to D$ where D is the unit disc such that $\phi(f^p(\phi^{-1}(z))) = \exp(2\pi i\theta)z$ for some irrational number θ .
- U is a Herman ring: there exists an holomorphic homeomorphism $\phi : U \to A$ where A is an annulus, $A = \{z : 1 < |z| < r\}, r > 1$ such that $\phi(f^p(\phi^{-1}(z))) = \exp(2\pi i\theta)z$ for some irrational number θ .
- U is an essentially parabolic or Baker domain: the boundary of U contains a point z_0, such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined.

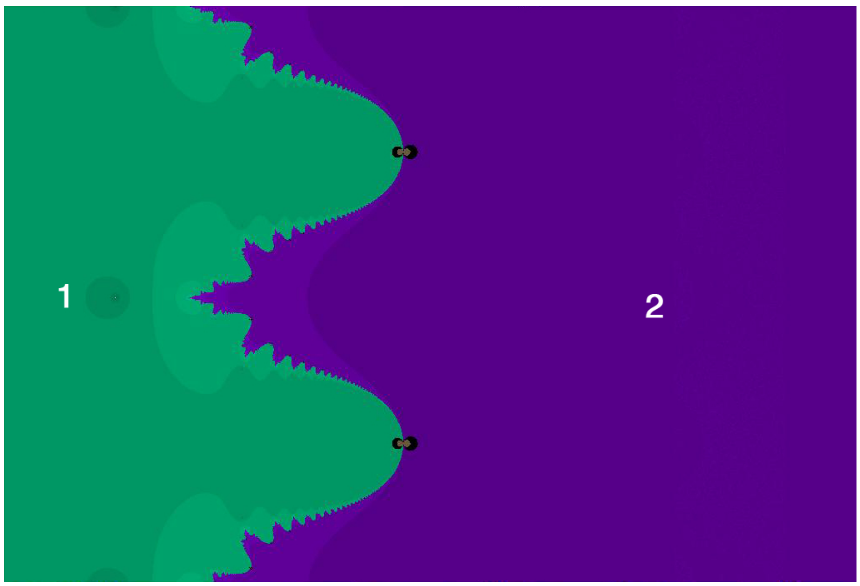

FIGURE 1. The Julia set of $T_{100,1}(z)$
The Julia set is the boundary curve of the green region (1) and the purple region (2).

3. The Dynamical planes of $T_a(z)$ and $E(z)$

In this section we discuss the dynamical planes for the functions $T_a(z)$ and for $E(z)$. These are the real slices of the families $T_{a,\lambda}$ and E_λ when $\lambda = 1$. Furthermore, we study the relationship between the two families. First let us describe the Julia set of the function for a fixed real $a > 0$ $T_a(z)$

Lemma 4. *For the family $T_a(z)$, when $0 < a < 1$, the function $T_a(z)$ has an attracting fixed point at $z = 0$ and two repelling fixed points on the real axis.*

When $a > 1$, there is a repelling fixed point at $z = 0$ and two attracting fixed points on the real axis with one in $a > Re(z) > \frac{1}{2}\log(a)$ and the other in $-1 < Re(z) < -\frac{1}{64}$.

When $a = 1$, there is a parabolic fixed point at $z = 0$.

Proof. First we observe that $z = 0$ is a fixed point of $T_a(z)$, for all a. By direct calculation, we see that when $0 < a < 1$,

$$|T_a'(0)| = \frac{2a}{a+1} < 1.$$

That is, $z=0$ is an attracting fixed point. When $a>1$,

$$|T'_a(0)| = \frac{2a}{a+1} > 1.$$

That is, $z=0$ is a repelling fixed point.
When $a>1$,

$$T_a(\frac{1}{2}\log(a)) - \frac{1}{2}\log(a) = \frac{a-1-\log(a)}{2} > 0$$

and

$$T_a(a) - a = \frac{a\exp(2a)-a}{\exp(2a)+a} - a = \frac{-a-a^2}{\exp(2a)+a} < 0.$$

Therefore, when $a > z > \frac{1}{2}\log(a)$, there exists a zero for the function $T_a(z) - z$, that is the fixed point z_0 of $T_a(z)$. Notice

$$|T'_a(z_0)| = |\frac{2a(1+a)}{(\exp(z_0)+a\exp(-z_0))^2}|$$

$$= |\frac{2a(a+1){z_0}^2}{a^2(\exp(z_0)-\exp(-z_0))^2}| = |\frac{2(a+1){z_0}^2}{(\exp(2z_0)+\exp(-2z_0)-2)a}|$$

$$< |\frac{2(a+1){z_0}^2}{a(4{z_0}^2)}| < \frac{a+1}{2a} < 1.$$

Here we used the fact that $\exp(z_0)+a\exp(-z_0) = \frac{a(\exp(z_0)-\exp(-z_0))}{z_0}$ and $\exp(2z)+\exp(-2z) = 1+2z+\frac{(2z)^2}{2!}+\cdots+1-2z+\frac{(-2z)^2}{2!}+\cdots > 2+(2z)^2$. Thus, z_0 is an attracting fixed point. Using a similar argument, and by direct calculation, we have $T_a(-\frac{1}{64})+\frac{1}{64} < 0$ and $T_a(-1)+1 > 0$. Therefore, when $-\frac{1}{64} > z > -1$, there exists a zero for the function $T_a(z)-z$, that is, a fixed point of $T_a(z)$. We can also estimate that $|T'_a(z)| < 1$, that is, it is an attracting fixed point.
When $a=1$, we have

$$T'_a(0) = 1.$$

That is, 0 is a parabolic fixed point. □

Proposition 2. *For the function $T_a(z)$, when $a>1$, the Julia set is a curve bounded by the vertical lines l_1 and l_2, where l_1 is the line $x=0$ and l_2 is the line $x=\frac{1}{2}\log(a)$. It passes through the points $z=n\pi i$ on l_1 and the points $z=\frac{1}{2}\log(a)+\frac{2n+1}{2}\pi i$ on l_2, for all n. The Fatou set consists of two completely invariant components each of which contains one of the asymptotic values.*

When $a=1$, the function $T_a(z)=\tanh(z)$ is conjugate to the tangent function $\tan(z)$ and the Julia set is the imaginary axis $x=0$.

When $0<a<1$, the Fatou set consists of one completely invariant component containing both asymptotic values and the Julia set of $T_a(z)$ is Cantor set.

Proof. When $a>1$, the function $T_a(z)$ maps the line l_1 into the left half plane H_L and maps H_L to a region to the left of $T_a(l_1)$. By Lemma 3.1, there is one attracting fixed point P_L in the left half plane and the other attracting fixed point P_R in the right half plane. By the Schwarz lemma, under iteration, the left half plane is attracted to P_L. The line l_2 is mapped to the line $x=\frac{a-1}{2}$ which is, since $a>1$, at the right of the line $x=\frac{1}{2}\log(a)$. Again by the Schwarz lemma , under iteration, the region $Re(z) > \frac{1}{2}\log(a)$ is attracted to the attracting fixed point P_R. The other two cases follow the arguments in the paper [KK1]. □

It is well known that the Julia set of the exponential function $E(z)$ is a Cantor bouquet. We give a brief explanation of this fact. The left half plane $x < 0$ is mapped to the disk $|z+1| < 1$ which eventually will be attracted to the attracting fixed point. The preimages of the left half plane form infinitely many finger-like regions. These infinitely many fingers are along the lines $x + \frac{1}{2}n\pi i$ for all n, one in each horizontal strip of width 2π. There are infinitely many narrower fingers which are mapped to each finger along the line $x + \frac{1}{2}n\pi i$. Thus the left half plane together with all the fingers are in the Fatou set. Their boundaries form a set of infinitely many curves all connected to ∞ in the Julia set.

We observe the following fact. When $a \to \infty$, the family $T_a(z)$ also converges uniformly on any compact subset to the exponential $E(z)$. Specifically, we have

Theorem 1. *Given any compact set $K \subset \mathbb{C}$, and any $\varepsilon > 0$, $\exists R > 0$ such that $\forall a > R$ and $z \in K$,*

$$|T_a(z) - (\exp(2z-1))| < \varepsilon.$$

Proof. A simple calculation shows that for all z, we have,

$$|T_a(z) - (exp(2z) - 1)| = |\frac{\exp(2z) - \exp(4z)}{\exp(2z) + a}|$$
$$\leq \frac{|\exp(4z)| + |\exp(2z)|}{|\exp(2z) + a|}$$

We can find a large enough, such that

$$K \subset \{z | Rez < \frac{1}{5}\log(a)\}.$$

For $z \in K$, we have,

$$|T_a(z) - (\exp(2z) - 1)| \leq \frac{|\exp(2z)| + |\exp(4z)|}{a - |\exp(2z)|}$$
$$= \frac{\exp(Re(2z)) + \exp(Re(4z))}{a - \exp(Re(2z))} < \frac{a^{\frac{2}{5}} + a^{\frac{4}{5}}}{a - a^{\frac{2}{5}}}.$$

The right hand side approaches 0 as $a \to \infty$. Therefore $\forall \varepsilon > 0$, we can find $R > 0$ such that when $a > R$, $\frac{a^{\frac{2}{5}} + a^{\frac{4}{5}}}{a - a^{\frac{2}{5}}} < \varepsilon$. □

In general, we have the following,

Theorem 2. *Given any compact set $K \subset \mathbb{C}$, given any $\lambda \neq 0$ and any $\varepsilon > 0$, $\exists R(\lambda) > 0$ such that $\forall a > R$ and $z \in K$,*

$$|T_{a,\lambda}(z) - E_\lambda(z)| < \varepsilon.$$

Proof. The proof of Theorem 2 is practically the same as the proof of Theorem 1. □

4. Symbolic Dynamics for $T_{a,\lambda}$ and E_λ

Now let us study the inverse branches of the function $T_{a,\lambda}$. For any fixed real number a, the function $T_{a,\lambda}(z)$ is periodic with period π and so is an infinite to one cover of $\hat{C} - \{-\lambda, a\lambda\}$. The origin is the fixed point and the points $z = n\pi i$ are the pre-images of the origin. The poles are $\frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$. The image of any vertical line segment between $\frac{1}{2}\log(a) + \frac{2n-1}{2}\pi i$ and $\frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ is the vertical line $x = \frac{a-1}{2}$. We denote by l_n the horizontal line $l_n = x + \frac{2n+1}{2}\pi i, x \in R, n \in Z$. We denote by L_n the horizontal strip between the lines l_{n-1} and l_n. Then the function $T_{a,\lambda}$ maps each horizontal strip L_n onto $\hat{C} - \{-\lambda, a\lambda\}$.

The inverse of $T_{a,\lambda}$ is given by the following multivalued formula:

$$T_{a,\lambda}^{-1}(z) = \frac{1}{2}\log(\frac{a\lambda + az}{a\lambda - z})$$

Let $\lambda = \lambda_x + i\lambda_y$, $z = x + iy$; then

$$Re(T_{a,\lambda}^{-1}(z)) = \frac{1}{2}\log|\frac{a\lambda + az}{a\lambda - z}| =$$

$$\frac{1}{4}\log(\frac{(a^2|\lambda|^2 - a|z|^2 + (a^2-a)(\lambda_x x + \lambda_y y))^2 + (a^2+a)^2(y\lambda_x - x\lambda_y)^2}{((a\lambda_x - x)^2 + (a\lambda_y - y)^2)^2}),$$

$$Im(T_{a,\lambda}^{-1}(z)) = \frac{1}{2}\arctan(\frac{(a^2+a)(y\lambda_x - x\lambda_y)}{a^2|\lambda|^2 - a|z|^2 + (a^2-a)(\lambda_x x + \lambda_y y)}).$$

In the second formula we must specify which branch of the arctan we use. We therefore denote by $T_{n,a,\lambda}^{-1}$ the branch of the inverse whose image is in the strip L_n. For a given $p \in \mathbb{N}$ and any sequence $\mathbf{n}_p = (n_1, n_2, n_3, \cdots, n_p)$, we define a branch of $T_{a,\lambda}^{-p}$ by

$$T_{a,\mathbf{n}_p,\lambda}^{-p} = T_{a,n_p,\lambda}^{-1} \circ T_{a,n_{p-1}\lambda}^{-1} \circ \cdots \circ T_{a,n_1,\lambda}^{-1}.$$

Definition 9. We call the sequence $\mathbf{n}_p$ the *itinerary* of the map $T_{a,\lambda}^{-p}$. We say an infinite sequence $\mathbf{n} = \{n_1, n_2, \cdots\}$ has bounded itinerary if there exists N, such that $|n_i| < N$ for all i.

When $\lambda > \frac{a+1}{2a}$, we have that 0 is a repelling fixed point of $T_{a,\lambda}$. There are two attracting fixed points P_L in the left half plane and P_R in the right half plane. Using the same argument as before, we know that the left half plane, $x < 0$, lies in the basin of P_L, and the right half plane, $x > \frac{1}{2}\log|a|$, lies in the basin of P_R. Now let us define the rectangles

$$R_n = \{z \in L_n | 0 \le Re(z) < \frac{1}{2}\log(a)\},$$

and set

$$\mathbf{R}_N = \cup R_i, |i| < N.$$

For any n, the inverse map $T_{a,\lambda,n}^{-1}$ is well-defined and analytic on any compact subset of R_n and takes values strictly inside the strip R_n. Hence $T_{a,n,\lambda}^{-1}$ is a strict contraction in the Poincare metric on R_n. As a consequence, for any infinite bounded sequence $\mathbf{n}$ with $|n_i| < N$, we can define the following:

$$T_{a,\mathbf{n},\lambda}^{-1} = \cdots \circ T_{a,n_p,\lambda}^{-1} \circ T_{a,n_{p-1},\lambda}^{-1} \circ \cdots \circ T_{a,n_1,\lambda}^{-1}(z).$$

For any $z \in \mathbf{R}_N$, $T_{a,\mathbf{n},\lambda}^{-1}$ is well-defined on $\mathbf{R}_N$.

Proposition 3. *Let $\mathbf{n} = n_1 n_2 \cdots$ be any bounded infinite sequence with $|n_i| < N$. Then there are some limit points $z_{a,\lambda}(\mathbf{n})$ in $\mathbf{R}_N$ whose itinerary under $T_{a,\lambda}$ is $\mathbf{n}$. These points lie in the Julia set of $T_{a,\lambda}$, where $\lambda > \frac{a+1}{2a}$. Moreover, if $\mathbf{n}$ is a repeating sequence, then $z_{a,\lambda}(\mathbf{n})$ is a repelling periodic point.*

Proof. From the above discussion we know for any k and any sequence $\mathbf{n}_k = n_1 n_2 \cdots n_k$, we have that the inverse map $F_k = T^{-1}_{a,n_k,\lambda} \circ T^{-1}_{a,n_{p-1}\lambda} \circ \cdots \circ T^{-1}_{a,n_1,\lambda}(z)$ is a contraction map in the region $\mathbf{R}_N$. The maps F_k are uniformly bounded and thus form normal family in the region $\mathbf{R}_N$. Therefore there exist limit functions for any subsequence F_{k_i}. Any limit function must be a constant by the Schwarz Lemma. When $n_1 n_2 \cdots n_{n_0} n_1 n_2 \cdots n_{n_0}, \cdots$ is a repeating sequence, then the maps of the subsequence $F_{n_0}, F_{2n_0}, \cdots, F_{kn_0}, \cdots$ are contractions in the region R_{n_0}. Thus this subsequence converges to an attracting fixed point z_0 of $T^{-n_0}_{a,\lambda}$ in the region R_{n_0}. Thus z_0 is a repelling fixed point of $T^{n_0}_{a,\lambda}$ and a repelling periodic point of $T_{a,\lambda}$. □

When $\lambda < \frac{a+1}{2a}$, 0 is the attracting fixed point of $T_{a,\lambda}$, and the real line is mapped to the interval $(-\lambda, a\lambda)$, which is attracted to the point 0. Since both asymptotic values are attracted to 0, the whole Fatou set is the basin of 0. The lines $x + n\pi i$ for all n are also mapped to the interval $(-\lambda, a\lambda)$. Thus these lines $l_n = x + n\pi i$ together with open strips L_n about them are in the basin of 0, that is, in the Fatou set. The far left half plane $H_L = \{z|Re(z) < -R\}$ is mapped to a small disk about $-\lambda$, so this half plane H_L lies in the basin of 0 as well. We can choose R large enough so that $T_{a,\lambda}(H_L) \subset L_0$. The far right half plane $H_R = \{z|Re(z) > R'\}$ is mapped to a small disk around the asymptotic value $a\lambda$. So this half plane also lies in the Fatou set. We can also choose R' large enough so that $T_{a,\lambda}(H_R) \subset L_0$. Let $\mathcal{O}$ be the union of these pieces of the basin. That is, $\mathcal{O} = \cup H_L \cup H_R \cup l_n$. So $T_{a,\lambda}(\mathcal{O}) \subset \mathcal{O}$. $\mathcal{O}$ is connected. Thus the complement of $\mathcal{O}$ consists of infinitely many closed, simply connected regions R_i. We have that $T_{a,\lambda}$ maps each R_i one to one onto $\hat{C} - T_{a,\lambda}(\mathcal{O})$, so $T_{a,\lambda}(R_i) \supset R_j \cup \{\infty\}$ for each j. There is a pole in each R_j. Thus for any given $\lambda \neq 0$ and $z \in J(T_{a,\lambda})$, we have that the orbit of z is contained in the $\cup R_i \cup \{\infty\}$ for all $i \in Z$. For any itinerary $\mathbf{n} = (n_1, n_2, \cdots)$, there exists at least one point z corresponding to the sequence $(n_1, n_2, \cdots)$. Thus we may use symbolic dynamics to associate to each $z \in J(T_{a,\lambda})$ an itinerary.

Definition 10. For any $z \in J(T_{a,\lambda})$, we can define the *itinerary* of z as one of the following two forms:

$$n(z) = n_0 n_1 n_2 \cdots \text{ or } n(z) = n_0 n_1 n_2 \cdots n_{n-1}\infty.$$

Here each $n_i \in Z$ and $n_i = k$ if and only if $T^i_{a,\lambda}(z) \in R_k$. If z is a prepole, $T^n_{a,\lambda}(z) = \infty$ and $T^{n-1}_{a,\lambda}(z)$ is a pole; we associate the finite sequence $n_0 n_1 n_2 \cdots n_{n-1}\infty$ to z.

Thus for any given λ and any point $z \in J(T_{a,\lambda})$, let Λ denote the set of all possible such itineraries, consisting of $(n_0, n_1, n_2, \cdots)$ where $n_j \in \mathbb{Z}$ and all finite sequences of the form $(n_0, n_1, n_2, \cdots n_j, \infty)$. There is a topology on Λ originally defined by Moser [Mo], by taking the usual neighborhood basis about an infinite itinerary. For a finite itinerary $\mathbf{n} = (n_1, n_2, \cdot, n_j, \infty)$, we associate a neighborhood basis of $\mathbf{n}$ to all (finite or infinite) sequences $n_0, n_1, \cdots, n_j\tau\cdots$ where $|\tau| \geq K$ for some $K \in \mathbb{Z}^+$. In this topology, Λ is homeomorphic to a Cantor set. There is a natural map called the shift automorphism $\sigma : \Lambda \to \Lambda$ defined by $\sigma(n_0, n_1, n_2, \cdots) = (n_1, n_2, \cdots)$. Note that $\sigma(\infty)$ is not defined. Let $\Lambda_{N,a,\lambda}$ be the set of all itineraries which remain in the region $\mathbf{R}_N$. Let Σ_N consist of all bounded itineraries $\mathbf{n} = (n_0, n_1, n_2, \cdots)$ with $|n_i| < N$.

The following theorem is for all maps in our family. The theorem is the same for tangent maps and exponential maps, as proven in [KK] and [Devaney] respectively.

Theorem 3. *Suppose $N > 0$. For each $\lambda \neq 0 \in \mathbf{C}$, $\Lambda_{N,a,\lambda}$ is homeomorphic to Σ_N and $T_{a,\lambda}|\Lambda_{N,a,\lambda}$ is conjugate to the shift map on Σ_N.*

Proof. We claim any point z in the Julia set is uniquely determined by its itinerary. If z is not a pole we use the argument from [McMullen] that T^n is expanding on the Julia set. We deduce that $z \in J(T_{a,\lambda})$ and ${T'_{a,\lambda}}^n(z) \to \infty$. At the poles we have $T'_{a,\lambda}(z) = \infty$. Therefore, $T_{a,\lambda}$ is expanding on its Julia set. Since two points corresponding to the same itinerary must remain a bounded distance apart, it follows that two points in the Julia set cannot have the same itinerary. Thus, we have the above result. □

Consequently, for any given itinerary $\mathbf{n} \in \Sigma_N$, we can define $z_{\lambda,a}(\mathbf{n})$ as the unique point in $\Lambda_{N,a,\lambda}$ under this homemorphism.

Now we set up the combinatorics for E_λ.

For the exponential family E_λ (see, [Devaney]), we can define horizontal strips $R(k) = R_\lambda(k)$ by

$$R(k) = \{z \in \mathbb{C} | (k - \frac{1}{2})\pi < Imz < (k + \frac{1}{2})\pi\}.$$

For any given $\lambda \neq 0$, we may use symbolic dynamics to associate to each $z \in J(E_\lambda)$ an itinerary of the following form:

$$\mathbf{n}(z) = n_0, n_1, n_2, \cdots$$

where $n_i = k$ if and only if $E^i_\lambda(z) \in R(k)$.

Now let us define the rectangle

$$R^b_0(n_i) = \{z \in R_{n_i} | 0 \leqslant Rez \leqslant b\} \text{ and } R^b_{0,K} = \cup_{|n_i| \leqslant K} R^b_0(n_i).$$

Throughout this section, we fix b such that

$$|\lambda|(\exp(2b) - 1) > K + |\lambda|.$$

Our choice of b guarantees that when $|n_i| < K$, $E_\lambda(R^b_0(n_i))$ covers each $R^b_0(n_j)$ for each $|n_j| < K$. Let $\Lambda_{K,\lambda}$ be the set of points whose orbits remain in R^b_0.

Definition 11. Σ_K is the set of itineraries $\{\mathbf{n} = (n_0, n_1, \cdots ,)\}$ such that $|n_i| < K$ for all $i \in Z$.

The following theorem is from [Devaney]

Theorem 4. *Suppose $K > 0$. For each $\lambda \in \mathbf{C}$, Σ_K is homeomorphic to $\Lambda_{K,\lambda}$ and $E_\lambda|\Lambda_{K,\lambda}$ is conjugate to the shift map on Σ_K.*

Thus for any given itinerary $\mathbf{n} \in \Sigma_K$, we can define $z_\lambda(\mathbf{n})$ to be the unique point in $\Lambda_{K,\lambda}$ under this homemorphism.

We have the following

Theorem 5. *For any given itinerary $\mathbf{n} \in \Sigma_K$, as $a \to \infty$ the corresponding points $z_{\lambda,a}(\mathbf{n}) \to z_\lambda(\mathbf{n})$.*

Proof. Let $\mathbf{n} = (n_0, n_1, n_2, \cdots)$ be any given bounded itinerary, and choose a $z \in R_T \cap R_b$. First, we claim that as $a \to \infty$, $T^{-1}_{a,\lambda,n_1}(z) \to E^{-1}_{\lambda,n_1}(z)$. This is clear since when $a \to \infty$, $T_{a,\lambda}(z)$ is uniformly convergent in the compact set to $E_\lambda(z)$. By induction, assume that when $i = k$, we have

$$T^{-1}_{a,\lambda,n_i} \circ \cdots \circ T^{-1}_{a,\lambda,n_1} \circ T^{-1}_{a,\lambda,n_0}(z) \to E^{-1}_{a,\lambda,n_i} \circ \cdots \circ E^{-1}_{\lambda,n_1} \circ E^{-1}_{\lambda,n_0}(z).$$

For any $\epsilon > 0$, there exists a R such that when $a > R$, $|z_{a,\lambda}(\mathbf{n_k}) - z_\lambda(\mathbf{n_k})| < \frac{1}{2}\epsilon$. When $i = k+1$, we have $T^{-1}_{a,\lambda,n_{k+1}} \to E^{-1}_{\lambda,n_{k+1}}$ and $|T^{-1}_{a,\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k})) - E^{-1}_{\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| < \frac{1}{2}\epsilon$ we can choose a large enough if necessary. Therefore $|T^{-1}_{a,\lambda,n_{k+1}}(z_{a,\lambda}(\mathbf{n_k})) - E^{-1}_{\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| < |T^{-1}_{a,\lambda,n_{k+1}}(z_{a,\lambda}(\mathbf{n_k})) - T^{-1}_{a,\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| + |T^{-1}_{a,\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k})) - E^{-1}_{\lambda,n_{k+1}}(z_\lambda(\mathbf{n_k}))| < \epsilon$.

Thus we have

$$\cdots \circ T^{-1}_{a,\lambda,n_2} \circ T^{-1}_{a,\lambda,n_1} \circ T^{-1}_{a,\lambda,n_0}(z) \to \cdots \circ E^{-1}_{a,\lambda,n_2} \circ E^{-1}_{\lambda,n_1} \circ E^{-1}_{\lambda,n_0}(z)$$

□

5. The Dynamic Planes of $T_{a,\lambda}$ and E_λ

Next we discuss the relationship between Julia sets of the function $T_{a,\lambda}$ when $\lambda > \frac{1}{2}$ and the exponential function E_λ. As we stated before, the Julia set can be characterized in two ways: the closure of the pre-poles and the closure of the repelling fixed points. First, when for any given $\lambda > \frac{1}{2}$, there exists a large k, such that when $a > k$, $z = 0$ is the repelling fixed point of $T_{a,\lambda}(z)$. The imaginary axis contains the pre-images of $z = 0$. Now let us consider the pre-images of the imaginary axis. By direct calculation, the first pre-image is,

$$\exp(4Re(z)) + (a-1)\exp(2Re(z))\cos(2Im(z)) - a = 0.$$

The curve passes through the points $z = n\pi i$ for all $n \in Z$ and the poles $\frac{1}{2}\log(a) + \frac{2n+1}{2}\pi i$ for all $n \in Z$. Let us denote this curve C_{T_1} and let $C^{i_1}_{T_1}$ be that part of the curve in R_i. That is,

$$C^{i_1}_{T_1} = \{\forall z \in C^{i_1}_{T_1} | z \in C_{T_1}, z \in R_{i_1}\}$$

Let $C^{i_1,i_2}_{T_2}$ be that part of the pre-image of the curve $C^{i_1}_{T_1}$ that lies in R_{i_2}. and let C_{T_2} be the pre-images of C_{T_1}, that is, $C_{T_2} = \cup C^{i_1,i_2}_{T_2}$ In general, we can follow this procedure; for each itinerary $\mathbf{i}_p = i_1, i_2, \cdots, i_p$, we can define

$$C^{i_1,i_2,\cdots,i_p}_{T_p} = T^{-1}_{a,i_p} \circ T^{-1}_{a,i_{p-1}} \circ \cdots \circ T^{-1}_{a,i_1}(z),$$

where $Re(z) = 0$ and $C^{i_1,i_2,\cdots,i_p}_{T_p} \in R_{i_p}$ and C_{T_p} is the pre-image of $C_{T_{p-1}}$.

By induction, we can show that the curve $C_{T_{k+1}}$ contains all of the pre-poles P_i up to $i = k$, where P_i is the pole of $T^i_a(z)$, and contains all of the pre-images of 0 under i^{th} iteration up to $i = k+1$. When $k \to \infty$, we claim that the curve $C_{T_k} = \cup C^{i_1,\cdots,i_k}_{T_k}$ converges to the Julia set of T_a. Clearly from the above discussion the limit curve C_{T_∞} contains the Julia set $J(T_a)$. On the other hand, assume that there is a point z is in the limit curve which is not a pre-pole. Then there exists a k and $z \in C_{T_k}$ with $T^k_a(z)$ on the imaginary axis. If $T^k_a(z)$ is not pre-image of 0, then $T^{k+1}_a(z)$ lies in the left half plane, and thus eventually maps to the attracting fixed point in the left half plane. Therefore, z is not in the curve $C_{T_{k+1}}$. This implies that z is not in the limit curve. Which is a contradiction. So, z must be in the Julia set.

We can use the same method to describe the Julia set of the exponential function $E(z)$. As above, we know that $z = 0$ is a repelling fixed point of $E(z)$ and the points $z = n\pi i$ are pre-images of $z = 0$. The pre-images of the imaginary axis will be the

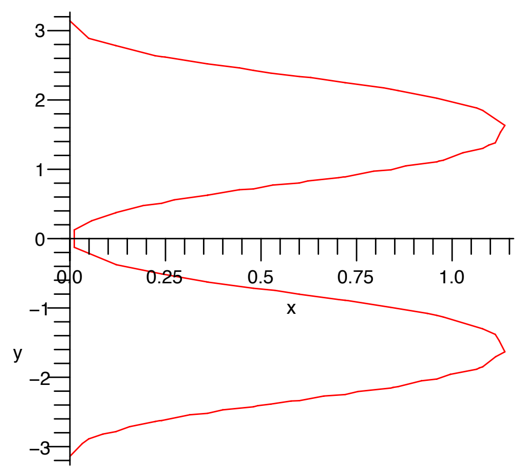

FIGURE 2. The inverse image of $T_{a,\lambda}$ when $a = 10$

union of curves, denoted by $\cup_{i_1=-\infty}^{\infty} C_1^{i_1}$ where $C_1^{i_1}$ is in the region

$$L_{i_1} = \{\forall z \in L_{i_1} | \frac{2i_1 - 1}{2} < Im(z) < \frac{2i_1 + 1}{2}\}.$$

In general, we denote

$$C_p^{i_1, i_2, \cdots, i_p} = E_{i_p}^{-1} \circ E_{i_{p-1}}^{-1} \circ \cdots \circ E_{i_1}^{-1}(z),$$

where $Re(z) = 0$ and $C_p^{i_1, i_2, \cdots, i_p} \in L_{i_p}$. Again as above, we can show that the limit curves will be in the Julia set of $E(z)$, which we saw in previous section is Cantor bouquet.

Theorem 6. *When $a > \frac{1}{2}$, 0 is a repelling fixed point of $T_{a,1}(z)$. For any given itinerary $\mathbf{i}_k = i_1, i_2, \cdots, i_k$, as $a \to \infty$ the curve $C_{T_k}^{i_1, \cdots, i_k}$ of $T_a(z)$ converges to the curve $C_k^{i_1, \cdots, i_k}$, where $C_{T_k}^{i_1, \cdots, i_k}$ and $C_k^{i_1, \cdots, i_k}$ are defined as above.*

Proof. We prove this by induction. When $n = 1$, from the calculation as we stated above, the curve $C_{T_1}^{i_1}$ is

$$\exp(4Re(z)) + (a - 1)\exp(2Re(z))\cos(2Im(z)) - a = 0$$

restricted in the region $\frac{2i_1 - 1}{2} < Im(z) < \frac{2i_1 + 1}{2}$, and the curve $C_1^{i_1}$ is

$$\exp(2Re(z))\cos(2Im(z)) - 1 = 0.$$

We have the curve

$$\exp(4Re(z)) + (a-1)\exp(2Re(z))\cos(2Im(z)) - a = 0.$$

Therefore

$$\cos(2Im(z)) = \frac{1 - \frac{1}{a}\exp(4Re(z))}{(1-\frac{1}{a})\exp(2Re(z))}$$

$$\to \frac{1}{\exp(2Re(z))} \text{ as } a \to \infty.$$

This shows $C_{T_1}^{i_1}$ goes to $C_1^{i_1}$

Assume that when $n = k$, the curve $C_{T_k}^{i_1,\cdots,i_k}$ approaches the curve $C_k^{i_1,\cdots,i_k}$. When $n = k+1$, $\forall z \in C_{T_{k+1}}^{i_1,\cdots,i_{k+1}}$, we have

$$T_a(z) = \zeta \in C_{T_k}^{i_1,\cdots,i_k}, \zeta = a\frac{\exp(2z)-1}{\exp(2z)+a}.$$

We also have

$$z = \frac{1}{2}\log(\frac{a\zeta + a}{a-\zeta})$$

where log takes the branch of $\frac{2i_{k+1}-1}{2} < Im(z) < \frac{2i_{k+1}+1}{2}$.

We know as $a \to \infty$,

$$Re(T_a^{-1}(z)) = \frac{1}{2}\log|\frac{a+az}{a-z}| = \frac{1}{4}\log(\frac{(a^2 - a|z|^2 + (a^2-a)x)^2 + (a^2+a)^2y^2}{((a-x)^2+y^2)^2})$$

$$= \frac{1}{4}\log(\frac{(1 - \frac{1}{a}|z|^2 + (1-\frac{1}{a})x)^2 + (1+\frac{1}{a})^2y^2}{((1-\frac{1}{a}x)^2 + \frac{1}{a}y^2)^2})$$

$$\to \frac{1}{4}\log((1+x)^2+1)^2 + y^2);$$

$$Im(T_a^{-1}(z)) = \frac{1}{2}\arctan(\frac{(a^2+a)y}{a^2 - a|z|^2 + (a^2-a)x})$$

$$\frac{1}{2}\arctan(\frac{(1+\frac{1}{a})y}{1 - \frac{1}{a}|z|^2 + (1-\frac{1}{a})x})$$

$$\to \frac{1}{2}\arctan(\frac{y}{1+x}).$$

Now we observe the inverse of $E^{-1}(z) = \frac{1}{2}\log(1+z)$. $Re(E^{-1}(z)) = \frac{1}{4}\log((x^2+1)^2 + y^2)$ and $Im(E^{-1}(z)) = \frac{1}{2}\arctan(\frac{y}{1+x})$. Therefore we have $\frac{1}{2}\log(\frac{a\zeta+a}{a-\zeta}) \to \frac{1}{2}\log(\zeta+1)$. By induction $\zeta \in C_{T_k}^{i_1,\cdots,i_k} \to$ some point $\zeta' \in C_k^{i_1,\cdots,i_k}$, and therefore, $\frac{1}{2}\log(\zeta+1)$ approaches $\frac{1}{2}\log(\zeta'+1)$ which is in the curve $C_{k+1}^{i_1,\cdots,i_{k+1}}$. □

From the above theorem, we have the following as an immediate corollary:

Theorem 7. *As $a \to \infty$, Julia set of $T_a(z)$ converges pointwise to the Julia set of $E(z)$.*

For any $z \in J(T_{a,\lambda})$ which is not prepole, there is a corresponding unique infinite itinerary and as $a \to \infty$, $z \to z_0$ which is the the end point of $J(E_\lambda)$.

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